Aspects of Supergravity in Eleven Dimensions

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Abstract

The scope of the present dissertation is to review some aspects of 11-dimensional supergravity. We describe the construction of the gravity supermultiplet and review the central charges that extend the $N = 1$ super-Poincaré algebra in eleven dimensions. We discuss the steps in the construction of the $d = 11$ supergravity Lagrangian, present the symmetries of the theory and the relevant transformation rules for the fields. We review the membrane and five-brane solutions and their properties: the saturation of the BPS bound by the relevant mass and charge densities, the preservation of half of the rigid space-time supersymmetries and the 'no-force-condition'. The supermembrane is introduced as the source that supports the space-time singularity of the membrane solution. We also discuss the interrelation between the saturation of the BPS bound and the partial breaking of supersymmetry and in the case of the membrane solution, the kappa symmetry of the supermembrane action. We review the Kaluza-Klein reduction of the 11-dimensional supergravity theory to ten dimensions. We describe the reduction of the bosonic sector of the action and show that the massless spectrum of the compactified theory coincides with the field content of type IIA supergravity in ten dimensions. In addition, we show how the string solution of $d = 10$ supergravity emerges upon a diagonal dimensional reduction of the membrane solution, while a double dimensional reduction of the membrane and the $d = 11$ supergravity background gives rise to a string coupled to type IIA supergravity.

Contents

Chapter 1

Introduction

Since its conception [1, 2], supergravity has played an important role in theoretical high-energy physics, merging the theory of general relativity with supersymmetry. In fact, supergravity arises as the *gauge theory of supersymmetry* [3]; the promotion of supersymmetry to a local symmetry signals the appearance of gravity [4, 6].

Consider the schematic form of the commutator of two supersymmetry transformations, with anticommuting parameters ϵ_1 and ϵ_2

$$[\delta(\epsilon_1), \delta(\epsilon_2)] \sim \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \, P_\mu \,, \tag{1.1}$$

where P_μ is the generator of space-time translations. As equation 1.1 shows, two successive supersymmetry transformations result in a space-time translation. Assume now that the parameters ϵ_1 and ϵ_2 depend on space-time points i.e. supersymmetry is converted from a global to a local symmetry. In this case, two consecutive supersymmetry transformations give rise to a local translation with parameter $\xi^\mu(x) = (\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x)$. On the other hand, local translations are the infinitesimal form of general coordinate transformations. Consequently, any theory that possesses local superymmetry invariance, must be invariant under general coordinate transformations as well. Accordingly, the metric appears as a dynamical field, rendering any locally supersymmetric theory, a theory of gravity.

Associated with any local (gauge) symmetry is a gauge field A_μ transforming as

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon + \ldots \,, \tag{1.2}$$

where ϵ is the parameter of the gauge transformation. In the case of local supersymmetry, the gauge transformation parameter is a spinorial object and so the relevant gauge field, known as *Rarita-Schwinger field*, carries one spinor and one vector index. The Rarita-Schwinger field, denoted $\Psi_\mu^\alpha(\mathrm{x})$, represents (on-shell) a particle of helicity 3/2, the *gravitino*. As the name implies, the gravitino is the superpartner of the graviton.

A supergravity theory in d dimensions is N*-extended*, when the number of supercharges that appear in the underlying supersymmetry algebra is $N \cdot n$, where n is the dimension of the minimal spinor representation in d space-time dimensions. A maximally extended supergravity theory has exactly thirty-two supersymmetries; the upper bound on the number of supersymmetries is based on the assumption that there are no physical fields with spin higher than two, the spin of the graviton field.

The reason why supergravity is appealing as a physical theory, is that supersymmetry imposes stringent constraints on its dynamics and field content, giving rise to rich mathematical structures [7]. Fields that typically appear in a gravity supermultiplet, apart from the graviton and N gravitinos, are p-form gauge fields, which are generalisations of the electromagnetic gauge potential and 'matter' fields such as scalar and spinor fields.

Supersymmetry has the property of alleviating the divergent ultraviolet behaviour of quantum field theories [9]. Thereupon, supergravity was originally conceived as a fundamental theory, capable of eliminating the non-renormalizable divergences that appear in the construction of a quantum field theory of gravity. Additionally, the particle content and symmetries of various supergravity models, made supergravity a viable framework for the unification of all fundamental forces [4, 6]. The current consensus is that although local supersymmetry improves the high-energy behaviour of quantum gravity, supergravity is an effective rather than a fundamental theory of nature.

Supergravity models provide fertile soil for phenomenological discussions in particle physics and cosmology [6, 8] but also play a prominent role in the context of string theory. The massless sector of the spectrum of superstring theories is described by supergravity [10] and thus by studying the behaviour of classical supergravity solutions, one retrieves valuable information about the low-energy dynamics of superstring theories. In addition, many results established at the supergravity level, such as dualities connecting different coupling regimes of various supergravity theories, can be elevated to the superstring level [28].

Extensively studied classical solutions of the supergravity field equations, are *brane solutions* [25, 26]. Brane solutions exhibit the structure of $(p+1)$-dimensional Poincaré-invariant hyperplanes, which are interpreted as world-volumes of objects extended in p spatial dimensions; these objects are known as *p-branes*. Brane solutions have a non-perturbative character and arise as electric, magnetic or dyonic [1] excitations of the $(p+1)$-form gauge fields that appear in supergravity theories. They are classified as *elementary* or *solitonic*, according to whether they are singular or non-singular solutions of the supergravity field equations.

A special class of brane solutions are *BPS brane solutions* [27]; these are supersymmetric solutions, characterised by the saturation of a Bogomol'nyi-Prasad-Sommerfield (BPS) bound which equates their mass density to the p-form charge(s) they carry. The BPS property 'shields' the brane solutions against quantum corrections [17] and thus, allows the extrapolation of results obtained in the classical limit, to the quantum level of string theory.

Among the various supergravity theories, 11-dimensional supergravity occupies a distinguished position; eleven is the maximal space-time dimension in which a supergravity theory can be constructed and possess no particle with helicity greater than two [19]. The supergravity theory in eleven dimensions was originally constructed [21] in order to obtain supergravity theories in lower dimensions, via *Kaluza-Klein dimensional reduction* [22, 44]. This approach eventually fell out of favour as it did not produce realistic models in four dimensions. The current perspective on 11-dimensional supergravity is that it describes the low-energy dynamics of *M-theory* [11]. At strong coupling, type IIA superstring theory on R^{10} is argued to coincide with M-theory on $\mathsf{R}^{10} \times \mathsf{S}^1$ [12]. The low-energy approximation of the former is type IIA supergravity, which arises upon a Kaluza-Klein reduction of 11-dimensional supergravity to ten dimensions. Accordingly, one expects that the supergravity theory in eleven dimensions is the low-energy effective field theory of M-theory.

The field content of $d = 11$ supergravity is rather simple: it comprises the graviton field, a Majorana gravitino field and a 3-form gauge field. Eleven-dimensional supergravity is a maximal supergravity theory and so the gravity supermultiplet is the unique $d = 11$, $N = 1$ supermultiplet; it is impossible to couple any independent 'matter' field. The field equations of 11-dimensional supergravity admit two BPS brane solutions: an *elementary membrane* solution and a *solitonic five-brane* solution, which arise as the electric and magnetic excitations of the 3-form gauge potential respectively.

[1] A dyon carries both electric and magnetic charges.

Chapter 2

The supersymmetry algebra in eleven dimensions

2.1 The gravity supermultiplet

The super-Poincaré algebra in eleven dimensions has the following form

$$[M_{MN}, M_{PQ}] = -\left(\eta_{MP}M_{NQ} - \eta_{MQ}M_{NP} - \eta_{NP}M_{MQ} + \eta_{NQ}M_{MP}\right) \tag{2.1a}$$

$$[P_M, P_N] = 0 \tag{2.1b}$$

$$[P_M, M_{PQ}] = \eta_{MP}P_Q - \eta_{MQ}P_P \tag{2.1c}$$

$$[Q_\alpha, M_{MN}] = (S_{MN})_\alpha{}^\beta\, Q_\beta, \qquad \alpha = 1, \ldots, 32 \tag{2.1d}$$

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta} P_M \tag{2.1e}$$

Relations 2.1a, 2.1b and 2.1c define the Poincaré algebra; M_{PQ} are the generators of $\mathsf{SO}(1,10)$ and P_M the generators of space-time translations. The supercharges Q_α transform as a single $\mathsf{Spin}(1,10)$ Majorana spinor (2.1d).

The gravity supermultiplet is a massless multiplet that contains a state of (highest) helicity 2, corresponding to the graviton and a state of helicity 3/2, corresponding to the gravitino. In the massless case the momentum 4-vector satisfies $P^2 = 0$ and in a fixed light-like reference frame $P_M = (-E, E, \ldots, 0)$. Inserting this expression into the anticommutator of the supercharges 2.1e gives

$$\{Q_\alpha, Q_\beta\} = E(-\Gamma^0 C^{-1} + \Gamma^1 C^{-1})_{\alpha\beta} = E(\mathbf{1} - \Gamma^{01})_{\alpha\beta}\,, \tag{2.2}$$

where we used $C^{-1} = \Gamma^0$ and defined $\Gamma^{01} \equiv \Gamma^{[0}\Gamma^{1]}$.

Since Γ^{01} squares to the identity and is traceless, half of its eigenvalues are $+1$ and half -1. Accordingly, the symmetric matrix $\{Q_\alpha, Q_\beta\}$ can be written as a diagonal matrix whose first sixteen entries equal $2E$ and the rest sixteen are zero. Supercharges that correspond to zero entries are discarded in the construction of the supermultiplet, as they produce non-physical states of zero norm. We conclude that only half of the supersymmetries are preserved and the supercharges $\tilde{Q}_\alpha \equiv (2E)^{-1/2} Q_\alpha$, where $\alpha = 1, \ldots, 16$, generate an $\mathsf{SO}(16)$ Clifford algebra

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = \delta_{\alpha\beta} \,. \tag{2.3}$$

We proceed to the construction of the gravity supermultiplet, by splitting the sixteen fermionic generators into eight operators that lower the helicity of a state by 1/2 and eight operators that raise the helicity of a state by 1/2 [5]. Introduce a Clifford vacuum state $|\Omega\rangle$ of lowest helicity -2 that is annihilated by the lowering operators; acting with the raising operators on $|\Omega\rangle$, yields the following tower of helicities

Helicity	*State*	*Degeneracy*
-2	$\|\Omega\rangle$	1
-3/2	$Q_i \|\Omega\rangle$	8
-1	$Q_i Q_j \|\Omega\rangle$	28
-1/2	$Q_i Q_j Q_k \|\Omega\rangle$	56
0	$Q_i Q_j Q_k Q_l \|\Omega\rangle$	70
1/2	$Q_i Q_j Q_k Q_l Q_m \|\Omega\rangle$	56
1	$Q_i Q_j Q_k Q_l Q_m Q_n \|\Omega\rangle$	28
3/2	$Q_i Q_j Q_k Q_l Q_m Q_n Q_p \|\Omega\rangle$	8
2	$Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7 Q_8 \|\Omega\rangle$	1

In order to match the above states to the fields that appear in the supergravity theory, it is necessary to find the helicity content of the latter. The helecity group is $\mathsf{SO}(2)$, the little group in four dimensions and so the helicity content of a field in eleven dimensions is unveiled by a dimensional reduction to four dimensions.

We start with the fields that we expect to appear in a supergravity theory, the graviton and the gravitino. The graviton is represented by a traceless symmetric tensor, whose decomposition under dimensional reduction is

$$h_{MN} \to h_{\mu\nu} + h_{\mu i} + h_{ij}\,, \qquad \mu,\nu = 0,\ldots,4 \qquad i,j = 1,\ldots,7 \tag{2.4}$$

giving rise to one state of helicity 2 ($h_{\mu\nu}$), seven states of helicity 1 ($h_{\mu i}$) and twenty-eight states of helicity 0 (h_{ij}). The gravitino decomposes as

$$\Psi^{\alpha}_{M} \to \Psi^{\alpha'\alpha''}_{\mu} + \Psi^{\alpha'\alpha''}_{i}, \tag{2.5}$$

where $\alpha' = 1,\ldots,4$ is the spinor index in four dimensions and $\alpha'' = 1,\ldots,8$ is the spinor index in seven dimensions. Hence, the reduction of the gravitino gives rise to eight states of helicity 3/2 ($\Psi^{\alpha'\alpha''}_{\mu}$) and fifty-six states of helicity 1/2 ($\Psi^{\alpha'\alpha''}_{i}$). Subtracting the graviton and gravitino states from the tower of helicities, we are left with twenty-one states of helicity 1 and forty-two states of helicity 0; these can be attributed to a 3-form field which decomposes as

$$A_{MNP} \to A_{\mu\nu\rho} + A_{\mu\nu i} + A_{\mu ij} + A_{ijk}. \tag{2.6}$$

giving rise to seven plus thirty-five states of helicity 0 ($A_{\mu\nu i} + A_{ijk}$) and twenty-one states of helicity 1 ($A_{\mu ij}$).

We conclude that the supergravity theory in eleven dimensions has a rather simple field content; it comprises a graviton, a gravitino and a 3-form field.

2.2 Central charges

The super-Poincaré algebra can be extended by additional generators known as *central charges* [15] that appear in the right-hand side of 2.1e

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta} P_M + \frac{1}{p!}\sum_p (\Gamma^{M_1\ldots M_p} C^{-1})_{\alpha\beta} Z_{M_1\ldots M_p}. \tag{2.7}$$

Here p is a non-negative integer and $\Gamma^{M_1\cdots M_p}$ denotes the antisymmetrised product of p gamma matrices $\Gamma^{[M_1}\Gamma^{M_2}\ldots\Gamma^{M_p]}$. The generators $Z^{M_1\cdots M_p}$ commute with P^M and Q_α and transform as rank-p antisymmetric tensors under Lorentz transformations.

Central charges typically appear in supersymmetric theories as topological charges of soliton solutions [16, 17]; in supergravity theories, p-form charges are associated with the gauge transformations of p-form fields and arise in the supersymmetry algebra realisation of BPS p-brane solutions [18].

Consistency with the symmetry properties of $\{Q_\alpha, Q_\beta\}$ requires that $(\Gamma_{M_1 \ldots M_p} C^{-1})_{\alpha\beta}$ is symmetric. Using B.1, B.3 and $C^t = -C$ one finds the following symmetry property for $\Gamma_{M_1 \ldots M_p} C^{-1}$ under transposition [28]

$$(\Gamma_{M_1 \ldots M_p} C^{-1})^t = (-1)^{\frac{(p-1)(p-2)}{2}} (\Gamma_{M_1 \ldots M_p} C^{-1}). \tag{2.8}$$

Equation 2.8 shows that $\Gamma_{M_1 \ldots M_p} C^{-1}$ is symmetric for $p = 1 \mod 4$ and $p = 2 \mod 4$. Due to the identity

$$\epsilon^{M_1 \cdots M_p M_{p+1} \cdots M_{11}} \Gamma_{M_{p+1} \cdots M_{11}} \propto \Gamma^{M_1 \cdots M_p}\,, \tag{2.9}$$

we need only to consider the cases $p = 2$ and $p = 5$. Hence, the anticommutator of the supercharges in the extended $d = 11$ super-Poincaré algebra is [20]

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta} P_M + \frac{1}{2!} (\Gamma^{M_1 M_2} C^{-1})_{\alpha\beta} Z_{M_1 M_2} + \frac{1}{5!} (\Gamma^{M_1 \ldots M_5} C^{-1})_{\alpha\beta} Z_{M_1 \ldots M_5}\,. \tag{2.10}$$

Notice that the left-hand side of the above equation has $32 \cdot 33/2 = 528$ components and so does the right-hand side: 11 components of P_M, 55 independent components of $Z_{M_1 M_2}$ and 462 independent components of $Z_{M_1 \ldots M_5}$. This means that the superalgebra is *maximally extended.*

The appearance of a p-form charge $Z_{M_1 \ldots M_p}$ in the supersymmetry algebra implies the existence of an object extended in p spatial dimensions i.e. a p-brane, which 'carries' the central charge. Since the fundamental fields of the supergravity theory do not carry such charges, the aforementioned p-brane is inherently *non-perturbative.* Accordingly, in the 11-dimensional supergravity theory we expect the presence of a 2-brane and a 5-brane.

Chapter 3

The supergravity theory in eleven dimensions

Theories that involve particles of spin higher than two are known to yield inconsistent interactions. Accordingly, the helicity of states that appear in the representations of a supersymmetry algebra should not exceed two; this requirement restricts the maximum number of supercharges [19] as follows. In the construction of a massless supermultiplet, only half of the initial supercharges contribute and half of the latter act as raising operators. Since each raising operator raises the helicity of a state by one-half, the highest number of raising operators one can have, without exceeding the helicity bound, is eight. Consequently, the maximum number of supercharges is $4 \cdot 8 = 32$.

The combination of the upper bound on the number of supercharges, with the dimension of the minimal spinor representation in d space-time dimensions leads to $d \leqslant 11$: in eleven dimensions a minimal spinor has exactly thirty-two components, while for $d \geqslant 12$ the dimension of the minimal spinor representation exceeds sixty-four (assuming only one time-like dimension) [14]. Therefore, eleven is the highest number of space-time dimensions in which a consistent supergravity theory exists.

In 1978 E. Cremmer, B. Julia and J. Scherk constructed the Lagrangian for the supergravity theory in eleven dimensions [21], in an attempt to obtain the $N = 8$ (maximally) extended supergravity theory in four dimensions, by dimensional reduction [22]. The method used in that construction was *Noether method*. Noether method is an iterative procedure for constructing a non-linear gauge theory, from the linear limit: the Lagrangian of the linearised theory is supplemented step by step with extra terms and the transformation rules of the fields are modified accordingly, until all variations vanish and a fully invariant Lagrangian is constructed [28].

Following the original paper, [21] we outline this procedure for the Lagrangian of $d = 11$ supergravity and present the symmetries of the theory.[1]

3.1 Construction of the Lagrangian

The gravity supermultiplet revealed the on-shell field content of the linearised $d = 11$ supergravity theory to consist of the graviton field (represented by) $e^{\alpha}{}_{\mu}(x)$, with 44 degrees of freedom, the gravitino field $\Psi_\mu(x)$, with 128 degrees of freedom and the 3-form gauge field $A_{\mu\nu\rho}(x)$, with 84 degrees of freedom. Thereupon, the starting point is a Lagrangian comprising the kinetic terms for the aforementioned fields

$$2\kappa_{11}^2 \mathcal{L} = eR(\omega) - 2ie\,\bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\omega)\Psi_\rho - \frac{e}{2\cdot 4!} F^{\mu_1\mu_2\mu_3\mu_4} F_{\mu_1\mu_2\mu_3\mu_4}\,, \tag{3.1}$$

where κ_{11} is the gravitational coupling constant and e the determinant of the vielbein. Furthermore, $F_{\mu_1\mu_2\mu_3\mu_4}$ is the field-strength of the gauge field $A_{\mu_1\mu_2\mu_3}$

$$F_{\mu_1\mu_2\mu_3\mu_4} = 4\,\partial_{[\mu_1} A_{\mu_2\mu_3\mu_4]} \tag{3.2}$$

and D_ν the covariant derivative of the gravitino field

$$D_\nu(\omega)\Psi_\mu = \partial_\nu \Psi_\mu + \frac{1}{4}\omega_\nu{}^{\alpha\beta}\Gamma_{\alpha\beta}\Psi_\mu\,. \tag{3.3}$$

The construction of the non-linear theory is performed in the one-half order formalism [4]; the equations of motion of the spin connection are assumed to be satisfied and so any terms arising from the variation of the Lagrangian with respect to the spin connection are neglected.

The initial supersymmetry transformation rules are

$$\delta e^{\alpha}{}_{\mu} = i\bar{\epsilon}\,\Gamma^{\alpha}\Psi_\mu\,, \quad \delta\Psi_\mu = D_\mu(\omega)\epsilon \quad \text{and} \quad \delta A_{\mu\nu\rho} = 0. \tag{3.4}$$

In the one-half formalism, the combined (linear) variation of the kinetic terms for the graviton and the gravitino fields vanishes [4] so we need only to compensate for the variation $\bar{\epsilon}\,\Psi F^2$ of

[1] We use a different metric convention and a different normalisation for the gauge and gravitino fields from the original paper.

the kinetic term for $A_{\mu\nu\rho}$. This is achieved by adding to the Lagrangian a term of the form $\bar{\Psi}X\Psi F$ and at the same time introducing a $Z\epsilon F$ term in the variation of the gravitino field

$$\delta\Psi_\mu = [D_\mu + (ZF)_\mu]\epsilon \,, \tag{3.5}$$

where X and Z are undetermined products of gamma matrices.

The easiest way to determine X and Z is to require the equations of motion of the gravitino field to be supercovariant i.e.

$$\Gamma^{\mu\nu\rho}\tilde{D}_\nu\Psi_\rho = 0 \,, \tag{3.6}$$

where $\tilde{D}_\nu$ is the supercovariant derivative defined by the variation of Ψ_ν: $\tilde{D}_\nu \equiv [D_\nu + (ZF)_\nu]$. Comparison of the terms that contain Z in 3.6 with the terms that contain X in the equations of motion derived by varying the $\bar{\Psi}X\Psi F$ terms in the action, fixes the form of X and Z and relates their coefficients.

Subsequently, a right adjustment of the undetermined coefficient of Z ensures that all terms of the form $\bar{\epsilon}\,\Psi F^2$ in the (altered) variation of the (modified) action vanish. The only exception is a term involving a product of nine gamma matrices, which is cancelled by supplementing the Lagrangian with a Chern-Simons term

$$a\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}} \,, \tag{3.7}$$

imposing the supersymmetry transformation rule

$$\delta A_{\mu\nu\rho} = b\,\bar{\epsilon}\,\Gamma_{[\mu\nu}\Psi_{\rho]} \tag{3.8}$$

and arranging appropriately the product ab. The coefficient b which is left undetermined, is fixed by requiring the terms of the form $\bar{\epsilon}\,\partial\Psi F$ and $\bar{\epsilon}\,\Psi\partial F$ in the variation of the action to vanish.

At this stage, all terms are fixed up to trilinear terms in $\delta\Psi$ and up to quartic terms in the gravitino field in the Lagrangian. The last step involves the replacement of F and ω in $\delta\Psi$ by their supercovariant counterparts

$$\hat{F}_{\mu_1\mu_2\mu_3\mu_4} = F_{\mu_1\mu_2\mu_3\mu_4} - 3\bar{\Psi}_{[\mu_1}\Gamma_{\mu_2\mu_3}\Psi_{\mu_4]} \quad \text{and} \quad \hat{\omega}_{\nu\alpha\beta} = \omega_{\nu\alpha\beta} - \frac{i}{4}\bar{\Psi}_\rho\Gamma_{\nu\alpha\beta}{}^{\rho\sigma}\Psi_\sigma \,, \tag{3.9}$$

where the spin connection ω is

$$\omega_{\nu\alpha\beta} = e^{\gamma}{}_{\nu}\left(\Omega_{\alpha\beta\gamma} - \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta}\right) + \frac{i}{4}\left[\bar{\Psi}_{\rho}\Gamma_{\nu\alpha\beta}{}^{\rho\sigma}\Psi_{\sigma} - 2\left(\bar{\Psi}_{\nu}\Gamma_{\beta}\Psi_{\alpha} - \bar{\Psi}_{\nu}\Gamma_{\alpha}\Psi_{\beta} + \bar{\Psi}_{\beta}\Gamma_{\nu}\Psi_{\alpha}\right)\right]. \tag{3.10}$$

Imposing the condition that the gravitino field equation obtained from the action is supercovariant, with respect to the aforementioned modified transformation $\delta\Psi$, fixes the quartic terms in the Lagrangian.

The final Lagrangian of the 11-dimensional supergravity theory is

$$\begin{aligned} 2\kappa_{11}^2\mathcal{L}_{11} = & eR(\omega) - \frac{e}{2\cdot 4!}F^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4} + \frac{1}{6\cdot 3!(4!)^2}\epsilon^{\mu_1\cdots\mu_{11}}A_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}} \\ & - 2ie\bar{\Psi}_{\mu}\Gamma^{\mu\nu\rho}D_{\nu}\left(\frac{\omega+\hat{\omega}}{2}\right)\Psi_{\rho} \\ & + \frac{ie}{96}\left(\bar{\Psi}_{\mu_1}\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\Psi_{\mu_2} + 12\bar{\Psi}^{\mu_3}\Gamma^{\mu_4\mu_5}\Psi_{\mu_6}\right)\left(F_{\mu_3\mu_4\mu_5\mu_6} + \hat{F}_{\mu_3\mu_4\mu_5\mu_6}\right) \end{aligned} \tag{3.11}$$

3.2 Symmetries and transformation rules

The symmetries of the Lagrangian and the corresponding transformation rules are (suppressing trivial transformations)

- General coordinate transformation with parameter ξ_μ

$$\delta e^{\alpha}{}_{\mu} = e^{\alpha}{}_{\nu}\partial_{\mu}\xi^{\nu} + \xi^{\nu}\partial_{\nu}e^{\alpha}{}_{\mu} \tag{3.12a}$$

$$\delta\Psi_{\mu} = \Psi_{\nu}\partial_{\mu}\xi^{\nu} + \xi^{\nu}\partial_{\nu}\Psi_{\mu} \tag{3.12b}$$

$$\delta A_{\mu_1\mu_2\mu_3} = 3A_{\rho[\mu_1\mu_2}\partial_{\mu_3]}\xi^{\rho} + \xi^{\rho}\partial_{\rho}A_{\mu_1\mu_2\mu_3} \tag{3.12c}$$

- Local $\mathsf{Spin}(1,10)$ transformations with parameter $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$

$$\delta e^{\alpha}{}_{\mu} = -e^{\beta}{}_{\mu}\lambda_{\beta}{}^{\alpha} \tag{3.13a}$$

$$\delta\Psi_{\mu} = -\frac{1}{4}\lambda_{\alpha\beta}\Gamma^{\alpha\beta}\Psi_{\mu} \tag{3.13b}$$

- $N = 1$ supersymmetry transformations with anticommuting parameter ϵ

$$\delta e^{\alpha}{}_{\mu} = i\bar{\epsilon}\,\Gamma^{\alpha}\Psi_{\mu} \tag{3.14a}$$

$$\delta\Psi_{\mu} = D_{\mu}(\hat{\omega})\epsilon - \frac{1}{12\cdot 4!}\,(\Gamma^{\nu_1\nu_2\nu_3\nu_4}{}_{\mu} + 8\,\Gamma^{\nu_1\nu_2\nu_3}\delta^{\nu_4}{}_{\mu})\,\hat{F}_{\nu_1\nu_2\nu_3\nu_4}\epsilon \tag{3.14b}$$

$$\delta A_{\mu_1\mu_2\mu_3} = 3i\bar{\epsilon}\,\Gamma_{[\mu_1\mu_2}\Psi_{\mu_3]} \tag{3.14c}$$

- Abelian gauge transformations with parameter $\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$

$$\delta A_{\mu_1\mu_2\mu_3} = \partial_{[\mu_1}\Lambda_{\mu_2\mu_3]} \tag{3.15}$$

- An odd number of space or time reflections together with $A_{\mu_1\mu_2\mu_3} \to -A_{\mu_1\mu_2\mu_3}$.

In addition, the transformation

$$e^{\alpha}{}_{\mu} \to e^{\sigma}e^{\alpha}{}_{\mu}\,, \quad \Psi_{\mu} \to e^{\sigma/2}\Psi_{\mu} \quad \text{and} \quad A_{\mu\nu\rho} \to e^{3\sigma}A_{\mu\nu\rho} \tag{3.16}$$

rescales the Lagrangian by a factor $e^{9\sigma}$, which can be absorbed in a redefinition of the gravitational coupling constant $\kappa_{11}^2 \to e^{9\sigma}\kappa_{11}^2$. Since the gravitational coupling constant is rescaled, the above transformation is not a symmetry of the action but it is a symmetry of the classical equations of motion.

Chapter 4

BPS branes in 11-dimensional supergravity

4.1 The membrane

4.1.1 The elementary membrane solution

The membrane solution of 11-dimensional supergravity was discovered in 1991 by M. J. Duff and K. S. Stelle [38], as a singular solution of the field equations that preserves half of the rigid space-time supersymmetries and saturates a BPS bound.

In constructing the membrane solution, we seek a bosonic configuration of the fields that reflects the presence of a membrane; a 2-dimensional object whose world-volume is a 3-dimensional hyperplane embedded in the 11-dimensional space-time. We also require that the solution preserve a fraction of the original space-time supersymmetries.

In the presence of a membrane, the initial Poincaré invariance in eleven dimensions is reduced to $\mathsf{P}_3 \times \mathsf{SO}(8)$ invariance i.e. Poincaré invariance in the flat world-volume of the membrane and rotational invariance in the transverse to the membrane directions. Accordingly, the membrane solution of the field equations should be invariant under the action of $\mathsf{P}_3 \times \mathsf{SO}(8)$.

We begin by making a three-eight split of the 11-dimensional space-time coordinates into 'world-volume' and 'transverse' coordinates respectively

$$x^M = (x^\mu, y^m)\,, \qquad \mu = 0,1,2 \qquad m = 3,\ldots,9\,. \tag{4.1}$$

The ansatz for the line element is

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}\delta_{mn}dx^m dx^n \tag{4.2}$$

and the associated vielbeins are

$$e^{\alpha}{}_{\mu} = e^{A}\,\delta^{\alpha}{}_{\mu} \quad \text{and} \quad e^{a}{}_{m} = e^{B}\,\delta^{a}{}_{m}\,. \tag{4.3}$$

Letters from the beginning of the alphabet are used for tangent-space indices. Since the 3-form gauge field naturally couples to the world-volume of the membrane, the relevant ansatz is

$$A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}\, e^{C}, \tag{4.4}$$

where $\epsilon_{\mu\nu\rho}$ is totally antisymmetric and $\epsilon_{012} = +1$. All other components of A_{MNP} are set to zero and so is the gravitino field Ψ_M. P_3 invariance requires that the arbitrary functions A, B and C depend only on y, while $\mathsf{SO}(8)$ invariance requires that this dependence be only through $r = \sqrt{\delta_{mn} y^m y^n}$.

If we require that the configuration of the fields be supersymmetric, the fields should be invariant under a supersymmetry transformation δ_ϵ with anticommuting parameter ϵ. Bosonic fields transform to fermionic ones and since the latter are set to zero, bosonic fields are invariant. The transformation rule for the gravitino field is $\delta_\epsilon \Psi_M|_{\Psi=0} = \tilde{D}_M \epsilon$, where henceforth $\tilde{D}_M$ denotes the supercovariant derivative

$$\tilde{D}_M|_{\Psi=0} = \partial_M + \frac{1}{4}\omega_M{}^{AB}\,\Gamma_{AB} - \frac{1}{12\cdot 4!}\left(\Gamma^{M_1M_2M_3M_4}{}_{M} + 8\,\Gamma^{M_1M_2M_3}\delta^{M_4}{}_{M}\right) F_{M_1M_2M_3M_4} \tag{4.5}$$

Consistency of setting the gravitino field to zero, with the assumption of residual supersymmetry, requires the existence of spinors ϵ, known as *Killing spinors*, satisfying

$$\tilde{D}_M \epsilon = 0\,. \tag{4.6}$$

We proceed to the solution of the Killing spinor equation, by adopting a basis for the gamma matrices compatible with the $\mathsf{P}_3 \times \mathsf{SO}(8)$ symmetry

$$\Gamma_A = \left(\gamma_\alpha \otimes \Sigma_9, \mathbf{1} \otimes \Sigma_a\right), \tag{4.7}$$

where γ_α are gamma matrices in $d = 3$ Minkowski space-time and Σ_a are gamma matrices in $d = 8$ Euclidean space. We have also defined $\Sigma_9 \equiv \Sigma_3\Sigma_4 \ldots \Sigma_{10}$ that satisfies $\Sigma_9^2 = \mathbf{1}$.

Furthermore, the spinor field $\epsilon(x,y)$ is decomposed as

$$\epsilon(x,y) = \epsilon_0 \otimes \eta(r)\,, \tag{4.8}$$

where ϵ_0 is a constant 2-component spinor of $\mathsf{Spin}(1,2)$ and $\eta(r)$ a 16-component spinor of $\mathsf{Spin}(8)$. The latter can be further decomposed into chiral eigenstates, by applying the projection operators $\frac{1}{2}(\mathbf{1} \pm \Sigma_9)$.

The next step in solving the Killing spinor equation is to evaluate $\tilde{D}_M$ for the ansätze 4.2 and 4.4. The spin connection $\omega_M{}^{AB}$ is expressed in terms of the vielbeins as in A.6. The $M = \mu$ component of the spin connection is

$$\begin{aligned}
\omega_\mu{}^{AB} &= e_{\gamma\mu}\left(\Omega^{AB\gamma} - \Omega^{B\gamma A} - \Omega^{\gamma AB}\right) \\
&= \frac{1}{2} e_{\gamma\mu}\left[(e^{A\nu}\, e^{Bm} - e^{B\nu}\, e^{Am})\partial_m e^\gamma{}_\nu + e^{\gamma\nu}\, e^{Bm}\,\partial_m e^A{}_\nu - e^{\gamma\nu}\, e^{Am}\,\partial_m e^B{}_\nu\right] \\
&= \frac{1}{2} e_{\gamma\mu}\, e^{-A}\left[(e^{A\nu}\, e^{Bm} - e^{B\nu}\, e^{Am})e^\gamma{}_\nu + e^{\gamma\nu}\, e^{Bm}\, e^A{}_\nu - e^{\gamma\nu}\, e^{Am}\, e^B{}_\nu\right]\partial_m e^A \\
&= e^{-A}\left(e^A{}_\mu\, e^{Bm} - e^B{}_\mu\, e^{Am}\right)\partial_m e^A\,.
\end{aligned}$$

Contracting $\omega_\mu{}^{AB}$ with Γ_{AB} gives

$$\omega_\mu{}^{AB}\,\Gamma_{AB} = 2\,e^{-A}\,\Gamma_\mu\Gamma^m\partial_m e^A = -2\,\gamma_\mu e^{-A}\,\Sigma^m\partial_m e^A\,\Sigma_9\,.$$

The terms in the supercovariant derivative $\tilde{D}_\mu$ that involve the field strength are

$$\Gamma^{M_1M_2M_3M_4}{}_\mu F_{M_1M_2M_3M_4} = 4\,\Gamma^{m\mu_2\mu_3\mu_4}{}_\mu\,\epsilon_{\mu_2\mu_3\mu_4}\,\partial_m e^C = 0$$

and

$$\begin{aligned}
\Gamma^{M_1M_2M_3}F_{M_1M_2M_3\mu} &= 3\,\Gamma^{m\mu_2\mu_3}\,\epsilon_{\mu_2\mu_3\mu}\,\partial_m e^C = 3\,\Sigma^m\gamma^{\alpha_2\alpha_3}\epsilon_{\alpha_2\alpha_3\alpha}\,e^\alpha{}_\mu\,\partial_m e^C \\
&= 3\cdot 2!\,\Sigma^m\gamma_\alpha\,e^{-3A}\,e^\alpha{}_\mu\,\partial_m e^C = 6\,\gamma_\mu e^{-3A}\Sigma^m\partial_m e^C\,.
\end{aligned}$$

Substituting the above expressions in 4.5 for $M = \mu$ we find

$$\tilde{D}_\mu = \partial_\mu - \frac{1}{2}\gamma_\mu e^{-A}\Sigma^m\partial_m e^A\Sigma_9 - \frac{1}{6}\gamma_\mu e^{-3A}\Sigma^m\partial_m e^C\,. \tag{4.9}$$

The $M = m$ component of the spin connection is

$$
\begin{aligned}
\omega_m{}^{AB} &= e_{cm}\left(\Omega^{ABc} - \Omega^{BcA} - \Omega^{cAB}\right) \\
&= \frac{1}{2}e_{cm}\left[(e^{Ak}e^{Bn} - e^{Bk}e^{An})\partial_n e^c{}_k - (e^{Bk}e^{cn} - e^{ck}e^{Bn})\partial_n e^A{}_k - (e^{ck}e^{An} - e^{Ak}e^{cn})\partial_n e^B{}_k\right] \\
&= \frac{1}{2}e_{cm}e^{-B}\left[(e^{Ak}e^{Bn} - e^{Bk}e^{An})e^c{}_k - (e^{Bk}e^{cn} - e^{ck}e^{Bn})e^A{}_k - (e^{ck}e^{An} - e^{Ak}e^{cn})e^B{}_k\right]\partial_n e^B \\
&= e^{-B}\left(e^A{}_m\, e^{Bn} - e^B{}_m\, e^{An}\right)\partial_n e^B\,.
\end{aligned}
$$

Contracting $\omega_m{}^{AB}$ with Γ_{AB} gives

$$
\omega_m{}^{AB}\,\Gamma_{AB} = 2\,e^{-B}\Sigma_m{}^n\,\partial_n e^B\,.
$$

The terms in the supercovariant derivative $\tilde{D}_m$ that involve the field strength are

$$
\begin{aligned}
\Gamma^{M_1M_2M_3M_4}{}_m F_{M_1M_2M_3M_4} &= 4\,\Gamma^{n\mu_2\mu_3\mu_4}{}_m\,\epsilon_{\mu_2\mu_3\mu_4}\,\partial_n e^C = 4\,\Sigma_m{}^n\Gamma^{\alpha_2\alpha_3\alpha_4}\epsilon_{\alpha_2\alpha_3\alpha_4}\,\partial_n e^C \\
&= 4\cdot 3!\,\Sigma_m{}^n\gamma^{012}\,\Sigma_9\,e^{-3A}\,\partial_n e^C = 24\,e^{-3A}\Sigma_m{}^n\,\partial_n e^C\Sigma_9\,,
\end{aligned}
$$

where we used the identity $\Gamma^{012} = \gamma^{012}\otimes\Sigma_9 = \mathbf{1}\otimes\Sigma_9$ and

$$
\begin{aligned}
\Gamma^{M_1M_2M_3}F_{M_1M_2M_3m} &= -\Gamma^{\mu_1\mu_2\mu_3}\,\epsilon_{\mu_1\mu_2\mu_3}\,\partial_m e^C = -\Gamma^{\alpha_1\alpha_2\alpha_3}\,\epsilon_{\alpha_1\alpha_2\alpha_3}\,\partial_m e^C \\
&= -3!\,\gamma^{012}\,\Sigma_9\,e^{-3A}\,\partial_m e^C = -6\,e^{-3A}\partial_m e^C\Sigma_9\,.
\end{aligned}
$$

Substituting the above expressions in 4.5 for $M = m$ we find

$$
\tilde{D}_m = \partial_m + \frac{1}{2}e^{-B}\Sigma_m{}^n\,\partial_n e^B - \frac{1}{12}e^{-3A}\Sigma_m{}^n\,\partial_n e^C\Sigma_9 + \frac{1}{6}e^{-3A}\partial_m e^C\Sigma_9\,. \tag{4.10}
$$

Upon substitution of the expressions 4.9, 4.10 and the decomposition 4.8 into the Killing spinor equation 4.6, the solution that arises is

$$
\epsilon = e^{C(r)/6}\epsilon_0\otimes\eta_0\,, \tag{4.11}
$$

where η_0 is a constant spinor satisfying

$$
(\mathbf{1} + \Sigma_9)\eta_0 = 0\,. \tag{4.12}
$$

In addition, A and B are determined in terms of C as

$$A = \frac{1}{3}C \quad \text{and} \quad B = -\frac{1}{6}C + constant\,. \tag{4.13}$$

The chirality condition 4.12 that the constant spinor η_0 satisfies, reduces its components by half. Consequently, the number of residual supesymmetries is $2 \cdot 8 = 16$ i.e. half of the original thirty-two rigid space-time supersymmetries. Also note that the requirement for residual supersymmetry results in the correlation of A, B and C, leaving only one function undetermined.

In order to determine C we turn to the Euler-Lagrange equations of the bosonic sector of the $d = 11$ supergravity action $I_{11}^{(b)} = \int d^{11}x\,\mathcal{L}_{11}^{(b)}$, where

$$2\kappa_{11}^2\,\mathcal{L}_{11}^{(b)} = \sqrt{-g}R(\omega) - \frac{\sqrt{-g}}{2\cdot 4!}F^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4} + \frac{1}{6\cdot 3!(4!)^2}\epsilon^{\mu_1\ldots\mu_{11}}A_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}} \tag{4.14}$$

The equation of motion of the 3-form gauge field is

$$\partial_M(\sqrt{-g}F^{MM_1M_2M_3}) + \frac{1}{2\cdot(4!)^2}\,\epsilon^{M_1M_2M_3\ldots M_{11}}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} = 0\,. \tag{4.15}$$

Inserting into 4.15 the ansätze 4.2, 4.4 and the expressions 4.13 gives

$$\begin{aligned}
\partial_m\left[e^{(3A+8B)(r)}\; g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}g^{mn}\epsilon_{\nu_1\nu_2\nu_3}\partial_n e^{C(r)}\right] &= 0 \\
\Rightarrow \partial_m\left[e^{(-3A+6B)(r)}\epsilon_{\nu_1\nu_2\nu_3}\delta^{mn}\partial_n e^{C(r)}\right] &= 0 \\
\Rightarrow \epsilon_{\nu_1\nu_2\nu_3}\delta^{mn}\partial_m\left[e^{-2C(r)}\partial_n e^{C(r)}\right] &= 0 \\
\Rightarrow \delta^{mn}\partial_m\partial_n e^{-C(r)} &= 0
\end{aligned} \tag{4.16}$$

i.e. a Laplace equation in the transverse directions. Imposing the boundary condition that the metric be asymptotically flat (Minkowski spacetime) yields the solution

$$e^{-C} = 1 + \frac{k_2}{r^6}\,, \quad r > 0\,. \tag{4.17}$$

where k_2 is an undetermined constant. The above expression also solves the Einstein equations.

Hence, the membrane solution consists of the line element

$$ds^2 = \left(1 + \frac{k_2}{r^6}\right)^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k_2}{r^6}\right)^{1/3} \delta_{mn} dx^m dx^n \tag{4.18}$$

and the 3-form gauge field

$$A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} \left(1 + \frac{k_2}{r^6}\right)^{-1} . \tag{4.19}$$

Consider the coordinate reparametrisation $r^6 = k_2[(1 - \tilde{r}^3)^{-1} - 1]$. In terms of $\tilde{r}$, the solution 4.18 becomes [45]

$$\begin{aligned} ds^2 = \tilde{r}^2(-dt^2 + d\sigma^2 + d\rho^2) + 4\, k_2^{1/3} \tilde{r}^{-2} d\tilde{r}^2 + k_2^{1/3} d\Omega_7^2 \\ + k_2^{1/3}[(1 - \tilde{r}^3)^{-1} - 1][4\, \tilde{r}^{-2} d\tilde{r}^2 + d\Omega_7^2] \, . \end{aligned} \tag{4.20}$$

Here we have introduced explicit coordinates $x^\mu = (t, \sigma, \rho)$ and $d\Omega_7^2$ is the line element of the unit 7-sphere, corresponding to the boundary of the 8-dimensional transverse space.

The geometry described by the line element 4.20 exhibits an event horizon at $\tilde{r} = 0$ and interpolates between two 'vacuum' solutions of 11-dimensional supergravity, corresponding to the limits $\tilde{r} \to 1$ i.e. transverse infinity and $\tilde{r} \to 0$ [45, 27].

- As $\tilde{r} \to 1$, the solution becomes asymptotically flat i.e. approaches Minkowski space-time.
- As $\tilde{r} \to 0$, one approaches the event horizon. The near horizon geometry is described by the first line of 4.20 which is the line element of $\mathsf{AdS}_4 \times \mathsf{S}^7$. The $\mathsf{AdS}_4 \times \mathsf{S}^7$ geometry is a stable solution of the supergravity field equations [44] arising from a *spontaneous compactification.* The spontaneous compactification is induced by a gauge field strength of the Freund-Rubin form $F_{\mu\nu\kappa\lambda} \propto \epsilon_{\mu\nu\kappa\lambda}$ [42].

Analytic continuation through the horizon reveals a curvature singularity at $\tilde{r} = -\infty$ [45]. The space-time singularity classifies the membrane solution as an *elementary solution* and requires the introduction of a δ-function source, so that equations of motion are satisfied everywhere; this is provided by the *supermembrane.*

4.1.2 The supermembrane

In 1987, E. Bergshoeff, E. Sezgin and P. K. Townsend [34] constructed the action for a supermembrane propagating in a $d = 11$ supergravity background. The key element of the action is a local fermionic symmetry on the supermembrane world-volume, known as *kappa symmetry.* A kappa symmetric supermembrane action requires the curved superspace background to obey certain constraints, including the existence of a closed super 4-form. These constraints are satisfied in the case of 11-dimensional supergravity and a consistent coupling is allowed [34].

The supermembrane is viewed as a bosonic manifold, embedded in $d = 11$ curved superspace; the corresponding coordinates are[1]

$$Z^{\bar{M}} = (X^M, \theta^\alpha)\,, \tag{4.21}$$

where X^M are the bosonic coordinates and θ^α the fermionic coordinates. The latter compose a 32-component $\mathsf{Spin}(1,10)$ Majorana spinor. The propagation of the supermembrane in the curved superspace background is described by a Green-Schwarz-type action [34]

$$\mathcal{S}_2 = T_2 \int d^3\xi \left(-\frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij} E^A{}_i E^B{}_j\, \eta_{AB} + \frac{1}{2}\sqrt{-\gamma} \right) + \frac{q_2}{3!} \int d^3\xi\, \epsilon^{ijk} E^{\bar{A}}{}_i E^{\bar{B}}{}_j E^{\bar{C}}{}_k\, A_{\bar{A}\bar{B}\bar{C}}\,. \tag{4.22}$$

Here $i = 0, 1, 2$ labels the coordinates ξ^i of the supermembrane world-volume with metric γ_{ij} and signature $(-, +, +)$. $A_{\bar{A}\bar{B}\bar{C}}$ is the super 3-form in the superspace description of 11-dimensional supergravity [23, 24] and $E^{\bar{A}}{}_i$ denotes the pull-back of the supervielbein $E^{\bar{A}}{}_{\bar{M}}$

$$E^{\bar{A}}{}_i \equiv (\partial_i Z^{\bar{M}}) E^{\bar{A}}{}_{\bar{M}}\,. \tag{4.23}$$

T_2 is the tension of the supermembrane and the term in the action that couples the super 3-form to the world-volume of the supermembrane is called *Wess-Zumino term.*

The symmetries of the action 4.22 are world-volume diffeomorphism invariance, target-space (superspace) superdiffeomorphism invariance, Lorentz invariance and super 3-form gauge invariance. In the case $q_2 = T_2$ (which is assumed henceforth) the supermembrane action possesses an additional local world-volume symmetry, kappa symmetry. The relevant transformation rules are [34]

$$\delta Z^{\bar{M}} E^A{}_{\bar{M}} = 0, \quad \delta Z^{\bar{M}} E^\alpha{}_{\bar{M}} = (\mathbf{1} - \Gamma)^\alpha{}_\beta\, \kappa^\beta(\xi) \tag{4.24}$$

[1] Henceforward, early Greek letters are used as spinorial indices and not as tangent-space indices.

and a variation of the worldvolume metric that is of no interest in our analysis. The kappa symmetry transformation parameter κ^{β} is a Majorana space-time spinor and a world-volume scalar. Γ is given by the expression

$$\Gamma = \frac{1}{3!\sqrt{-\gamma}}\,\epsilon^{ijk} E^{A}{}_{i} E^{B}{}_{j} E^{C}{}_{k}\,\Gamma_{ABC}\,. \tag{4.25}$$

Kappa (or κ-) symmetry originally appeared as a symmetry of the superparticle [31] and the superstring [32] and provides the link between space-time and world-volume supersymmetry [30]. In 1986, it was generalised for higher dimensional objects (branes) by J. Hughes, J. Liu and J. Polchinski [33], who explicitly constructed a Green-Schwarz-type action for a super 3-brane in six dimensions.

Since we are considering a bosonic configuration, the fermionic coordinates θ^{α} are set to zero and the supermembrane action reduces to

$$S_2 = T_2 \int d^3\xi \left(-\frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M \partial_j X^N g_{MN} + \frac{1}{2}\sqrt{-\gamma} + \frac{1}{3!}\,\epsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P A_{MNP} \right). \tag{4.26}$$

Variation of the membrane action 4.26 with respect to X^M yields the equation of motion

$$\partial_i\left(\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^N g_{MN}\right) + \frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^N \partial_j X^P \partial_M g_{NP} = -\frac{1}{3!}\,\epsilon^{ijk}\partial_i X^N \partial_j X^P \partial_k X^Q F_{MNPQ}\,, \tag{4.27}$$

while varying with respect to γ_{ij} gives

$$\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN} \tag{4.28}$$

i.e. γ_{ij} is identified with the induced from the target space world-volume metric.

Having set θ^{α} to zero, the criterion for residual supersymmetry is that under a combination of a κ-symmetry and a space-time supersymmetry transformation, θ^{α} remain zero. The variation of the fermionic coordinates to linear order in fermion is [36]

$$\delta\theta = (\mathbf{1} - \Gamma)\kappa + \epsilon\,, \tag{4.29}$$

where κ is the κ-symmetry parameter and ϵ the space-time supersymmetry parameter. The bosonic part of Γ is given by the expression

$$\Gamma = \frac{1}{3!\sqrt{-\gamma}}\,\epsilon^{ijk}\partial_i X^M \partial_i X^N \partial_i X^P\,\Gamma_{MNP}\,. \tag{4.30}$$

Γ is traceless and has the property $\Gamma^2 = \mathbf{1}$. We prove the latter.

$$\Gamma^2 = \frac{1}{36\gamma}\,\epsilon^{ijk}\,\epsilon_{lmn}\,\partial_i X^{M_1}\partial_j X^{M_2}\partial_k X^{M_3}\,\partial^l X_{N_1}\partial^m X_{N_2}\partial^n X_{N_3}\,\Gamma_{M_1M_2M_3}\,\Gamma^{N_1N_2N_3}\,. \tag{4.31}$$

The product of gamma matrices that appears in the last expression is expanded as [14]

$$\Gamma_{M_1M_2M_3}\Gamma^{N_1N_2N_3} = 6\,\delta_{[M_3}{}^{[N_1}\delta_{M_2}{}^{N_2}\delta_{M_1]}{}^{N_3]} + 18\,\delta_{[M_3}{}^{[N_1}\delta_{M_2]}{}^{N_2]}\Gamma_{M_1}{}^{N_3} + 9\,\delta_{M_3}{}^{N_1}\Gamma_{M_1M_2}{}^{N_2N_3} \tag{4.32}$$

and so we find

$$\begin{aligned}\Gamma^2 =& \frac{1}{36\gamma}\Big[6\,\epsilon^{ijk}\epsilon_{ijk} + 18\,\epsilon^{ijk}\epsilon_{kjn}\,\partial_i X^{M_1}\,\partial^n X_{N_3}\,\Gamma_{M_1}{}^{N_3} \\ &\qquad + 9\,\epsilon^{ijk}\epsilon_{kmn}\,\partial_i X^{M_1}\partial_j X^{M_2}\,\partial^m X_{N_2}\partial^n X_{N_3}\,\Gamma_{M_1M_2}{}^{N_2N_3}\Big] \\ =& \frac{1}{36\gamma}6\,\epsilon^{ijk}\epsilon_{ijk} = \mathbf{1}\,,\end{aligned} \tag{4.33}$$

in courtesy of the embedding equation 4.28.

Due to the aforementioned properties of Γ, the matrices $(\mathbf{1}\pm\Gamma)$ act as projection operators. Consequently, κ-symmetry can be used to set sixteen of the thirty-two components of θ^α to zero, by imposing $(\mathbf{1}-\Gamma)\theta = 0$. Around a purely bosonic background the above condition is preserved if $(\mathbf{1}-\Gamma)\delta\theta = 0$ [36] or upon inserting the variation 4.29

$$2\,(\mathbf{1}-\Gamma)\kappa + (\mathbf{1}-\Gamma)\,\epsilon = 0\,. \tag{4.34}$$

The above equation implies that $\kappa = -\epsilon/2$ and thus

$$\delta\theta = \frac{1}{2}(\mathbf{1}+\Gamma)\epsilon\,. \tag{4.35}$$

Therefore, the requirement for residual supersymmetry boils down to $(\mathbf{1}+\Gamma)\epsilon = 0$.

The reparametrisation invariance on the world-volume of the supermembrane allows one to impose a physical or light-cone gauge condition [35]. The fully gauged-fixed world-volume theory is *globally supersymmetric*; it contains eight bosonic degrees of freedom, corresponding to the embedding scalars and eight fermionic degrees of freedom. The former appear as the Goldstone bosons associated with broken translation invariance in the transverse dimensions and the latter as the Goldstinos associated with broken rigid supersymmetry [33]. The origin of world-volume supersymmetry is the kappa invariance of the supermembrane action, which can be used to gauge away half of the fermionic coordinates.

4.1.3 The combined supergravity-supermembrane equations

As we saw earlier, the membrane solution has a singularity at the core, requiring the introduction of a δ-function source for its support. We therefore supplement the supergravity action $I_{11}^{(b)}$ with the bosonic sector 4.26 of the supermembrane action.

The Einstein equation in the presence of the membrane becomes [38]

$$R_{MN} - \frac{1}{2} g_{MN} R = \kappa_{11}^2 T_{MN} \, , \tag{4.36}$$

where the stress-energy tensor $T^{MN} = T^{MN}_{(1)} + T^{MN}_{(2)}$ receives a contribution from the 3-form gauge field

$$T^{MN}_{(1)} = \frac{1}{12\kappa_{11}^2} \left(F^M{}_{PQR} F^{NPQR} - \frac{1}{8} g^{MN} F_{PQRS} F^{PQRS} \right) \tag{4.37}$$

and the membrane

$$T^{MN}_{(2)} = -T_2 \int d^3\xi \sqrt{-\gamma}\, \gamma^{ij} \partial_i X^M \partial_j X^N \, \frac{\delta^{11}(x-X)}{\sqrt{-g}} \, . \tag{4.38}$$

Moreover, the equation of motion of the 3-form gauge potential is

$$\partial_M \left(\sqrt{-g} F^{M M_1 M_1 M_3} \right) + \frac{1}{2 \cdot (4!)^2} \epsilon^{M_1 M_2 M_3 M_4 \dots M_{11}} F_{M_4 M_5 M_6 M_7} F_{M_8 M_9 M_{10} M_{11}} = -2\kappa_{11}^2 T_2 \int d^3\xi \, \epsilon^{ijk} \, \partial_i X^{M_1} \partial_j X^{M_2} \partial_k X^{M_3} \, \delta^{11}(x-X) \tag{4.39}$$

or more elegantly

$$\mathbf{d}(\star \mathbf{F}^{(4)} + \tfrac{1}{2} \mathbf{A}^{(3)} \wedge \mathbf{F}^{(4)}) = -2\kappa_{11}^2 \star \mathbf{J}^{(3)}, \tag{4.40}$$

where $\star$ denotes the Hodge dual [56]. The above equation gives rise to a conserved Page charge [48]

$$Q_2 = \frac{1}{\sqrt{2}\kappa_{11}} \int_{\partial\Sigma_8} (\star \mathbf{F}^{(4)} + \tfrac{1}{2} \mathbf{A}^{(3)} \wedge \mathbf{F}^{(4)}) \tag{4.41}$$

where $\partial\Sigma_8$ is the boundary of the 8-dimensional transverse space Σ_8. Equivalently, the conserved charge can be expressed as

$$Q_2 = \sqrt{2}\kappa_{11} \int_{\Sigma_8} \star \mathbf{J}^{(3)} \tag{4.42}$$

where

$$J^{M_1 M_2 M_3}(x) = T_2 \int d^3\xi \, \epsilon^{ijk} \, \partial_i X^{M_1} \partial_j X^{M_2} \partial_k X^{M_3} \, \delta^3(x - X) \, . \tag{4.43}$$

Q_2 is conserved by virtue of the equations of motion of the 3-form gauge potential and thus it is an 'electric' charge.

The combined supergravity-supermembrane equations are solved by the static gauge choice [38]

$$X^\mu = \xi^\mu \, , \qquad \mu = 0, 1, 2 \tag{4.44}$$

in which the $p + 1$ longitudinal space-time coordinates are identified with the coordinates of the membrane world-volume and the solution

$$X^m = Y^m = constant, \qquad m = 3, \ldots, 10 \tag{4.45}$$

Substituting 4.44 and 4.45 into 4.39 yields the equation

$$\delta^{mn} \partial_m \partial_n e^{-C(r)} = -2\kappa_{11}^2 T_2 \, \delta^8(y) \, . \tag{4.46}$$

Integrating both sides over the volume of the 8-dimensional transverse space Σ_8 gives

$$\int_{\Sigma_8} d^8x \, \partial^m \partial_m e^{-C} = -2\kappa_{11}^2 T_2 \, . \tag{4.47}$$

Since the geometry of the membrane solution is asymptotically flat, the integral on the left-hand side of the above equation can be transformed to a surface integral over the boundary $\partial\Sigma_8$ of the transverse space.

$$\int_{\Sigma_8} d^8x \, \partial^m \partial_m e^{-C} = \int_{\partial\Sigma_8} d\Sigma^m \partial_m e^{-C} \tag{4.48}$$

Rotational invariance in the transverse directions allows a 7-sphere S^7 boundary. Hence,

$$\int_{\partial\Sigma_8} d\Sigma^m \partial_m e^{-C} = \int_{\mathsf{S}^7} d\Sigma^m \partial_m \left[1 + \frac{k_2}{r^6} + \mathcal{O}(r^{-12})\right] = \int_{\mathsf{S}^7} d\Omega_7 \, r^6 y^m \, \frac{-6k_2 y_m}{r^8} = -6k_2\Omega_7 \, , \tag{4.49}$$

where Ω_7 is the volume of the unit 7-sphere. Therefore, k_2 is determined in terms of the membrane tension T_2 as

$$k_2 = \frac{\kappa_{11}^2 T_2}{3\Omega_7}. \tag{4.50}$$

A consistent coupling of the membrane to the supergravity background of the membrane solution, requires that the former preserve one-half of the original rigid space-time supersymmetries, as does the latter. We saw in the previous section that this is the case if $(\mathbf{1}+\Gamma)\epsilon = 0$. For the solution 4.44 and 4.45

$$\Gamma = \mathbf{1} \otimes \Sigma_9 \tag{4.51}$$

and by virtue of $(\mathbf{1} + \Sigma_9)\eta_0 = 0$, the required condition $\Gamma\epsilon = \epsilon$ is satisfied.

4.1.4 The BPS property and the 'no force' condition

The asymptotically flat geometry of the membrane solution allows the definition of an ADM [46] mass density $\mathcal{M}_2$: a conserved mass per unit spatial surface of the membrane [37, 47].

We start by making the split $g_{MN} = \eta_{MN} + h_{MN}$, where η_{MN} is the flat Minkowski metric; for the membrane solution h_{MN} tends to zero at spatial infinity. The part of the Einstein tensor that is linear in h_{MN} is regarded as the 'kinetic' term and the remaining as the gravitational stress-energy pseudo-tensor. Therefore, the 'world-volume' component of the latter is given by the expression

$$\Theta_{\mu\nu} = \frac{1}{\kappa_{11}^2}\left(R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}\right). \tag{4.52}$$

The linearised Ricci tensor $R^{(1)}_{\mu\nu}$ is given in terms of h_{MN} as

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 h^P{}_\nu}{\partial x^P \partial x^\mu} + \frac{\partial^2 h^P{}_\mu}{\partial x^P \partial x^\nu} - \frac{\partial^2 h^P{}_P}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial x_P \partial x^P}\right) , \tag{4.53}$$

where indices are raised and lowered using the flat Minkowski metric. Inserting the metric ansatz 4.2 into the above equation gives

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\frac{\partial^2 h_{\mu\nu}}{\partial y_m \partial y^m} = -\frac{1}{2}\eta_{\mu\nu}\frac{\partial^2 e^{2A}}{\partial y_m \partial y^m} \tag{4.54}$$

Similarly for the Ricci scalar $R^{(1)} \equiv g^{MN} R^{(1)}_{MN}$ we get

$$R^{(1)} = \frac{\partial^2 h^P{}_M}{\partial x^P \partial x^M} - \frac{\partial^2 h^M{}_M}{\partial x_P \partial x^P} = -3\frac{\partial^2 e^{2A}}{\partial y_m \partial y^m} - (8-1)\frac{\partial^2 e^{2B}}{\partial y_m \partial y^m} \tag{4.55}$$

Inserting 4.54 and 4.55 into 4.52 we find

$$\Theta_{\mu\nu} = \frac{1}{2\kappa_{11}^2}\eta_{\mu\nu}\left[2\frac{\partial^2 e^{2A}}{\partial y^2} + 7\frac{\partial^2 e^{2B}}{\partial y^2}\right]. \tag{4.56}$$

The total energy density of the membrane solution is then given by an integral of the Θ_{00} component of the stress-energy pseudo-tensor, over the volume Σ_8 of the transverse space

$$\mathcal{E} = \int_{\Sigma_8} d^8x\, \Theta_{00} = -\frac{1}{2\kappa_{11}^2}\int_{\Sigma_8} d^8x\left[2\frac{\partial^2 e^{2A}}{\partial y^2} + 7\frac{\partial^2 e^{2B}}{\partial y^2}\right]. \tag{4.57}$$

The above integral can be recast as a surface integral over the boundary $\partial\Sigma_8$

$$\mathcal{E} = -\frac{1}{2\kappa_{11}^2}\int_{\partial\Sigma_8} d\Sigma^m \partial_m\left(2e^{2A} + 7e^{2B}\right). \tag{4.58}$$

Recall that the existence of residual supersymmetry imposed the relations 4.13. Hence 4.58 becomes

$$\mathcal{E} = -\frac{1}{2\kappa_{11}^2}\int_{\partial\Sigma_8} d\Sigma^m \partial_m\left(\tfrac{4}{3}e^C - \tfrac{7}{3}e^C\right) = \frac{1}{2\kappa_{11}^2}\int_{\partial\Sigma_8} d\Sigma^m \partial_m e^C = \frac{6k_2\Omega_7}{2\kappa_{11}^2}, \tag{4.59}$$

where the surface integral is evaluated similar to 4.49. Since we are considering the membrane in its rest frame, the energy density $\mathcal{E}$ coincides with the mass density $\mathcal{M}_2$ of the membrane.

On the other hand, the charge density of the membrane solution is

$$\begin{aligned} Q_2 &= \frac{1}{\sqrt{2}\kappa_{11}}\int_{\partial\Sigma_8}\left(\star\mathbf{F}^{(4)} + \tfrac{1}{2}\mathbf{A}^{(3)}\wedge\mathbf{F}^{(4)}\right) \\ &= \frac{1}{\sqrt{2}\kappa_{11}}\int_{\partial\Sigma_8} d\Sigma^{m_1\dots m_7}\,\epsilon_{m_1\dots m_7 m}\,\partial^m e^C = \frac{1}{\sqrt{2}\kappa_{11}}\int_{\partial\Sigma_8} d\Sigma^m \partial_m e^C = \frac{6k_2\Omega_7}{\sqrt{2}\kappa_{11}} \end{aligned} \tag{4.60}$$

Comparing equations 4.58 and 4.60 we conclude that

$$Q_2 = \sqrt{2}\kappa_{11}\mathcal{M}_2 \tag{4.61}$$

i.e. the membrane solution saturates the BPS bound $\sqrt{2}\kappa_{11}\mathcal{M}_2 \geqslant Q_2$. In addition, using 4.50 we find

$$Q_2 = \sqrt{2}\kappa_{11}\mathcal{M}_2 = \sqrt{2}\kappa_{11}T_2\,. \tag{4.62}$$

The connection between the partial breaking of supersymmetry and the saturation of the BPS bound is also revealed in the asymptotic realisation of the supersymmetry algebra [13]. The geometry of the membrane solution at spatial infinity is Poincaré invariant and admits Killing spinors. Accordingly, the supersymmetry algebra is realised asymptotically; the supercharges and the momentum 4-vector are defined as Noether charges of the asymptotic symmetries by ADM-type formulas [49, 37]

$$P^M = \int_{\Sigma_{10}} d^{10}x\,\Theta^{0M} \tag{4.63}$$

$$Q^\alpha = \int_{\partial\Sigma_{10}} d\Sigma_{NP}(\Gamma^{MNP}\Psi_M)^\alpha\,, \tag{4.64}$$

where Σ_{10} is a 10-dimensional space-like surface.

As it turns out [18], the asymptotic supersymmetry algebra is not the super-Poincaré algebra but the extended version

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta}P_M + \frac{1}{2!}(\Gamma^{M_1M_2}C^{-1})_{\alpha\beta}Z_{M_1M_2}. \tag{4.65}$$

The origin of the modification of the supersymmetry algebra is the term in the supermembrane action 4.22 that describes the coupling of the super 3-form to the supermembrane world-volume; under a supersymmetry transformation, the variation of the Wess-Zumino term produces a total derivative [18]. In general, when a Lagrangian $\mathcal{L}(\phi^m,\, \partial_i\phi^m)$ is quasi-invariant i.e. it transforms by a total derivative

$$\delta_a\mathcal{L} = \partial_i\Delta_a^i \tag{4.66}$$

the conserved currents J_a^i contain an anomalous piece Δ_a^i

$$J_a^i = \frac{\partial \mathcal{L}}{\partial(\partial_i \phi^m)} \delta_a \phi^m - \Delta_a^i \tag{4.67}$$

and the charge densities J_a^0 satisfy a modified Poisson bracket algebra. In the case of the supermembrane the Wess-Zumino term gives rise to a central charge [18]

$$Z^{M_1 M_2} = T_2 \int d\xi_1 d\xi_2 \, j^{0 M_1 M_2}, \tag{4.68}$$

where $j^{0M_1M_2}$ is the world-volume time component of the identically conserved topological current

$$j^{iM_1M_2} = \epsilon^{ijk} \partial_i X^{M_1} \partial_j X^{M_2}. \tag{4.69}$$

Consider a stationary membrane along the x_1-x_2 plane so that $P_M = (-\mathcal{M}_2 V_2, 0, \ldots, 0)$, where V_2 is the spatial surface of the membrane[2] and assume a static gauge. The supersymmetry algebra 4.65 becomes

$$\{Q_\alpha, Q_\beta\} = \mathcal{M}_2 V_2 \cdot \mathbf{1}_{\alpha\beta} + (\Gamma^{012})_{\alpha\beta} Z_{12} \, . \tag{4.70}$$

The Majorana spinor Q_α is real and so the left-hand side of the above equation is positive definite. Hence $\mathcal{M}_2$ must satisfy the bound

$$\mathcal{M}_2 V_2 \geqslant |Z_{12}| \, . \tag{4.71}$$

Comparison of the definition 4.42 of the electric charge density and the definition 4.68 of the topological charge yields $\sqrt{2}\kappa_{11}|Z_{12}| = Q_2 V_2$ and so the familiar form of the BPS bound $\sqrt{2}\kappa_{11}\mathcal{M}_2 \geqslant Q_2$ is retrieved.

In the case of the membrane we are considering, equation 4.68 yields

$$Z_{12} = T_2 \int d\xi_1 d\xi_2 = T_2 V_2 = \mathcal{M}_2 V_2 \, . \tag{4.72}$$

[2] V_2 is a formal factor which should be normalised, so that the supersymmetry algebra is well-defined per unit spatial surface of the membrane.

As a consequence, the BPS bound is saturated and 4.70 becomes

$$\{Q_\alpha, Q_\beta\} = \mathcal{M}_2 V_2 \, (\mathbf{1} + \Gamma^{012})_{\alpha\beta}. \tag{4.73}$$

Since Γ^{012} squares to the identity and is traceless, the matrix $(\mathbf{1}+\Gamma^{012})$ acts like a projection operator, which projects out half of the supercharges. Hence, the fact that only half of the original thirty-two rigid supersymmetries are preserved by the membrane solution, is reproduced in the realisation of the supersymmetry algebra.

Recall that initially the Wess-Zumino term appeared in the supermembrane action 4.22 multiplied by a q_2 factor. The same factor instead of T_2 would appear in the topological charge 4.68 if the requirement for kappa invariance of the action did not dictate $q_2 = T_2$. In the latter case, as the above analysis revealed, the BPS bound is saturated and a partial breaking of supersymmetry is induced. Thus, the BPS property and the partial breaking of supersymmetry are traced back to the kappa symmetry of the supermembrane action.

Another property of the membrane solution related to the existence of residual supersymmetry is the 'no-force' condition [37, 38]: the exact cancellation of attractive gravitational and repulsive electrostatic (generated by the 3-form gauge field) forces, between two separated static membranes.

Consider a stationary test membrane at some distance from a source membrane located at the origin. Both membranes run along the x_1-x_2 plane and have the same orientation. The motion of the test membrane is described by the action 4.26 in the background of the source membrane, upon substitution of 4.28. Assuming the static gauge $X^\mu = \xi^\mu$, the potential acting on the stationary test membrane is

$$V = -\sqrt{\det\left(-\eta_{ij}\, e^{2A}\right)} + A_{012} = -e^{3A} + e^{C}\,. \tag{4.74}$$

The existence of Killing spinors imposed $3A = C$ and so $V = 0$; the stationary test membrane experiences no force.

The 'no-force' condition allows the construction of stable multi-membrane configurations obtained by a linear superposition of solutions [38]

$$e^{-C} = 1 + \sum_i \frac{k_2}{|\mathbf{r} - \mathbf{r}_i|}\,, \tag{4.75}$$

where $\mathbf{r}_i$ corresponds to the arbitrary location of each membrane.

4.2 The five-brane

As anticipated from the presence of a 5-form central charge in the supersymmetry algebra, there is one more BPS brane solution of the $d = 11$ supergravity equations, the *5-brane solution* [39]. In contrast to the membrane, the 5-brane is a *soliton* i.e. non-singular solution of the field equations and arises as the 'magnetic' excitation of the 3-form gauge field. Due to the non-singular nature of the 5-brane, a source is not required in the solution and one does not need to include a σ-model term in the supergravity action.

The construction of a bosonic field configuration with residual supersymmetry, that describes a 5-brane, mirrors the construction of the membrane solution. In the case of the 5-brane the invariance group is $\mathsf{P}_6 \times \mathsf{SO}(5)$. Accordingly, we make a six-five split of the space-time coordinates

$$x^M = (x^\mu, y^m)\,, \qquad \mu = 0, \ldots, 5 \qquad m = 6, \ldots, 10. \tag{4.76}$$

The ansatz for the line element is

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}\delta_{mn}dx^m dx^n. \tag{4.77}$$

The 5-brane is a 'magnetic' excitation of the gauge potential and so the corresponding field strength supports the transverse to the 5-brane space. The relevant ansatz is

$$F_{mnpq} = -\epsilon_{mnpqr}\,\partial^r e^{-C}. \tag{4.78}$$

All other components of F_{MNPQ} are set to zero and so is the gravitino field Ψ_M. P_6 invariance requires that the arbitrary functions A, B and C depend only on y, while $\mathsf{SO}(5)$ invariance requires that this dependence be only through $r = \sqrt{\delta_{mn}y^m y^n}$.

As we argued in the case of the membrane solution, the requirement for residual supersymmetry is equivalent to solving the Killing spinor equation $\tilde{D}_M\epsilon = 0$. We adopt a basis for the gamma matrices compatible with the $\mathsf{P}_6 \times \mathsf{SO}(5)$ symmetry

$$\Gamma_A = (\gamma_\alpha \otimes \mathbf{1}, \gamma_7 \otimes \Sigma_a)\,, \tag{4.79}$$

where γ_α are gamma matrices in $d = 6$ Minkowski space-time and Σ_a are gamma matrices in $d = 5$ Euclidean space. We have also defined $\gamma_7 \equiv \gamma_0\gamma_1 \dots \gamma_5$ that satisfies $\gamma_7^2 = \mathbf{1}$. Furthermore, the spinor field $\epsilon(x, y)$ is decomposed as

$$\epsilon(x, y) = \epsilon_0 \otimes \eta(r)\,, \tag{4.80}$$

where $\eta(r)$ is a 4-component spinor of $\mathsf{Spin}(5)$ and ϵ_0 is a constant 8-component spinor of $\mathsf{Spin}(1, 5)$. The latter can be further decomposed into chiral eigenstates, by applying the projection operators $\frac{1}{2}(\mathbf{1} \pm \gamma_7)$.

In the background 4.77 and 4.78, the components of the supercovariant derivative 4.5 are

$$\tilde{D}_\mu = \partial_\mu + \frac{1}{2}\gamma_\mu e^{-A}\Sigma^m\partial_m e^A\gamma_7 + \frac{1}{12}\gamma_\mu e^{-3B}\Sigma^m\partial_m e^{-C} \tag{4.81a}$$

$$\tilde{D}_m = \partial_m + \frac{1}{2}e^{-B}\Sigma_m{}^n\,\partial_n e^B + \frac{1}{12}e^{-3B}\partial_m e^{-C}\gamma_7 - \frac{1}{6}e^{-3B}\Sigma_m{}^n\,\partial_n e^{-C}\gamma_7\,. \tag{4.81b}$$

Substituting the decomposition 4.80 and the above expressions for the supercovariant derivative, into the Killing spinor equation 4.6, yields the solution

$$\epsilon = e^{C(r)/12}\epsilon_0 \otimes \eta_0\,, \tag{4.82}$$

where η_0 is a constant spinor and ϵ_0 satisfies $(\mathbf{1} - \gamma_7)\epsilon_0 = 0$. Additionally, A and B are determined in terms of C as

$$A = \frac{1}{6}C \quad \text{and} \quad B = -\frac{1}{3}C + constant\,. \tag{4.83}$$

The chirality condition that the constant spinor ϵ_0 satisfies, reduces its components by half. Consequently, the number of residual supesymmetries is $4 \cdot 4 = 16$; the 5-brane solution preserves half of the original rigid space-time supersymmetries.

Upon substitution of 4.77, 4.78 and 4.83, the Einstein equation and the equation of motion of the 3-form gauge field reduce to the single equation

$$\delta^{mn}\partial_m\partial_n e^{-C(r)} = 0. \tag{4.84}$$

Imposing the boundary condition that the geometry be asymptotically flat yields the solution

$$e^{-C} = 1 + \frac{k_5}{r^3}\,, \quad r > 0\,. \tag{4.85}$$

Hence, the 5-brane solution consists of the line element

$$ds^2 = \left(1 + \frac{k_5}{r^3}\right)^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k_5}{r^3}\right)^{2/3} \delta_{mn} dx^m dx^n \tag{4.86}$$

and the 4-form field strength

$$F_{mnpq} = 3k_5\, \epsilon_{mnpqr} \frac{y^r}{r^5}. \tag{4.87}$$

The 5-brane geometry exhibits an event horizon but no curvature singularity. Furthermore, it interpolates between two 'vacuum' solutions of $d = 11$ supergravity; Minkowski space-time at spatial infinity and $\mathsf{AdS}_7 \times \mathsf{S}_4$ [43] space-time near the horizon [27].

The ADM mass density $\mathcal{M}_5$ of the 5-brane solution is evaluated similar to $\mathcal{M}_2$

$$\mathcal{M}_5 = \frac{\eta_{00}}{2\kappa_{11}^2} \int_{\Sigma_5} d^5x \left[5\frac{\partial^2 e^{2A}}{\partial y^2} + 4\frac{\partial^2 e^{2B}}{\partial y^2} \right] = \frac{3k_5 \Omega_4}{2\kappa_{11}^2}\,, \tag{4.88}$$

where Ω_4 is the volume of the unit 4-sphere corresponding to the boundary of the transverse space. Moreover, the 5-brane solution carries a 'magnetic' charge

$$P_5 = \frac{1}{\sqrt{2}\kappa_{11}} \int_{\partial\Sigma_5} \mathbf{F}^{(4)}, \tag{4.89}$$

where $\partial\Sigma_5$ is the boundary of the 5-dimensional transverse space Σ_5. P_5 is conserved by virtue of the Bianchi identity $\mathbf{dF}^{(4)} = 0$. Using 4.87 we find

$$P_5 = \frac{1}{\sqrt{2}\kappa_{11}} \int_{\partial\Sigma_5} d\Sigma^{m_1 \ldots m_4} \epsilon_{m_1 \ldots m_4 m} \frac{3k_5 y^m}{r^5} = \frac{3k_5}{\sqrt{2}\kappa_{11}} \int_{\mathsf{S}^4} d\Omega_4\, r^3\, \frac{y_m y^m}{r^5} = \frac{3k_5\Omega_4}{\sqrt{2}\kappa_{11}}. \tag{4.90}$$

Therefore, $P_5 = \sqrt{2}\kappa_{11}\mathcal{M}_5$ and the 5-brane solution saturates the relevant BPS bound.

The electric charge of the membrane and the magnetic charge of the 5-brane obey a Dirac quantization rule

$$Q_2\, P_5 = 2\pi n\,, \quad n \in \mathbb{Z} \tag{4.91}$$

and so P_5 can be expressed in terms of the membrane tension as

$$P_5 = \frac{2\pi n}{\sqrt{2}\kappa_{11} T_2} \,. \tag{4.92}$$

A 'no-force' condition, related to the existence of residual supersymmetry, applies to the 5-brane solution and allows for stable composite configurations of membranes and 5-branes [40, 41].

Chapter 5

Kaluza-Klein reduction to ten dimensions

5.1 Dimensional reduction of d = 11 supergravity over a circle

A maximal supergravity theory in ten space-time dimensions can be of type IIA or type IIB, depending on the chiral character of the underlying superalgebra [7]. The minimal spinor representation of $\mathsf{Spin}(1,9)$ is a 16-dimensional Majorana-Weyl representation [14]. Accordingly, the thirty-two supercharges of the $d = 10$, $N = 2$ supersymmetry algebra compose two Majorana-Weyl spinors; if these two spinors are of opposite chirality, they can be assembled into a single Majorana spinor and the resulting non-chiral supersymmetry algebra underpins type IIA supergravity. On the other hand, type IIB supergravity is based on a chiral supersymmetry algebra: the supercharges compose two Majorana-Weyl spinors of the same chirality.

Reduction of the $d = 11$, $N = 1$ supersymmetry algebra to ten dimensions, yields the non-chiral $d = 10$, $N = 2$ supersymmetry algebra, as a $\mathsf{Spin}(1,10)$ Majorana spinor decomposes to two $\mathsf{Spin}(1,9)$ Majorana-Weyl spinors of opposite chirality [55]. The above reduction elevates to the supergravity theories: type IIA supergravity in ten dimensions is obtained as the massless spectrum of $d = 11$ supergravity with one dimension compactified on a circle [51, 52, 53].

Assume that the eleventh dimension of $d = 11$ space-time has circular (S^1) topology, i.e. the x^{10} coordinate, denoted henceforth as ρ, is periodic

$$0 \leq \rho \leq 2\pi R\,, \tag{5.1}$$

where R denotes the radius of the circle. Let $\phi(x,\rho)$ be a field in eleven dimensions. Due to the periodicity of ρ, $\phi(x,\rho)$ can be expanded as

$$\phi(x,\rho) = \phi(x) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} \exp\left(\frac{in\rho}{R}\right)\phi_n(x)\,. \tag{5.2}$$

In the limit $R \to 0$ only the $n = 0$ modes survive and the 10-dimensional theory comprises fields which do not depend on the compactified dimension.

We proceed to the dimensional reduction of the bosonic sector of $d = 11$ supergravity in the $R \to 0$ limit and make a $10 + 1$ split of the vielbein. Local Lorentz invariance in eleven dimensions allows for a triangular parametrisation [50]

$$\hat{e}^{\hat{\alpha}}{}_{\hat{\mu}} = \begin{pmatrix} e^{\delta\Phi}\, e^{\alpha}{}_{\mu} & 0 \\ e^{\Phi} A_{\mu} & e^{\Phi} \end{pmatrix} \qquad \hat{\mu},\hat{\alpha} = 0,\ldots,10 \qquad \mu,\alpha = 0,\ldots,9\,. \tag{5.3}$$

Henceforward, a hat symbol designates an object in eleven dimensions. The exponential parametrisation of the scalar field is imposed to ensure positivity and δ is a parameter that will be determined by requiring the Einstein-Hilbert term in the reduced Lagrangian to have a canonical form. The inverse of the vielbein 5.3 is

$$\hat{e}^{\hat{\mu}}{}_{\hat{\alpha}} = \begin{pmatrix} e^{-\delta\Phi}\, e^{\mu}{}_{\alpha} & 0 \\ -e^{-\delta\Phi} A_{a} & e^{-\Phi} \end{pmatrix}. \tag{5.4}$$

The corresponding space-time metric in eleven dimensions is given by

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\delta\Phi} g_{\mu\nu} + e^{2\Phi} A_{\mu} A_{\nu} & e^{2\Phi} A_{\nu} \\ e^{2\Phi} A_{\mu} & e^{2\Phi} \end{pmatrix}. \tag{5.5}$$

As the above expression shows, the metric decomposition yields the 10-dimensional space-time metric $g_{\mu\nu}$, a vector field A_{μ} and a scalar field e^{Φ}. In order to prove this statement in a concrete way, we turn to the general coordinate transformation

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{\xi}^{\hat{\rho}}\hat{g}_{\hat{\rho}\hat{\nu}} + \partial_{\hat{\nu}}\hat{\xi}^{\hat{\rho}}\hat{g}_{\hat{\rho}\hat{\mu}} + \hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} \tag{5.6}$$

and assume that the parameters $\hat{\xi}^{\hat{\mu}}$ are independent of the compactified dimension ρ. The

$\hat{\mu} = \mu$ component of 5.6 yields

$$\delta g_{\mu\nu} = \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\rho\mu} + \xi^\rho \partial_\rho g_{\mu\nu} \tag{5.7a}$$

$$\delta A_\mu = \partial_\mu \xi^\rho A_\rho + \partial^\rho \partial_\rho A_\mu \tag{5.7b}$$

$$\delta \Phi = \xi^\rho \partial_\rho \Phi. \tag{5.7c}$$

The above transformation rules in ten dimensions validate the scalar and vector character of e^Φ and A_μ respectively. Furthermore, the general coordinate transformation with parameter $\hat{\xi}^{10}$ acts as a local gauge transformation for A_μ

$$\delta A_\mu = \partial_\mu \hat{\xi}^{10}. \tag{5.8}$$

Let us now focus on the reduction of the Einstein-Hilbert term in the $d = 11$ supergravity action

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \, \hat{e} \, \hat{R}(\hat{\omega}) \tag{5.9}$$

or in the language of differential forms

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2} \int \hat{\mathbf{R}}_{\hat{\alpha}\hat{\beta}} \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}) = \frac{1}{2\kappa_{11}^2} \int (\mathbf{d}\hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\beta}} + \hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\gamma}} \wedge \hat{\boldsymbol{\omega}}^{\hat{\gamma}}{}_{\hat{\beta}}) \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}), \tag{5.10}$$

where $\hat{\mathbf{R}}_{\hat{\alpha}\hat{\beta}}$ is the curvature form of the spin connection. The term in the Einstein-Hilbert action involving the exterior derivative of the spin connection can be recast as

$$\mathbf{d}\hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\beta}} \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}) = \mathbf{d}[\, \hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\beta}} \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}})\,] + \hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\beta}} \wedge \mathbf{d} \star (\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}})\,. \tag{5.11}$$

Due to the vanishing torsion condition $\mathbf{D}\hat{\mathbf{e}}^{\hat{\alpha}} = 0$,

$$\mathbf{D} \star (\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}) = 0 = \mathbf{d} \star (\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}) + \hat{\boldsymbol{\omega}}^{\hat{\alpha}\hat{\gamma}} \wedge \star(\hat{\mathbf{e}}_{\hat{\gamma}} \wedge \hat{\mathbf{e}}^{\hat{\beta}}) + \hat{\boldsymbol{\omega}}^{\hat{\beta}\hat{\gamma}} \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}_{\hat{\gamma}}). \tag{5.12}$$

Discarding the total derivative terms and using the above identity to replace $\mathbf{d} \star (\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}})$, we arrive at the following expression for the Einstein-Hilbert action

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2} \int \hat{\boldsymbol{\omega}}_{\hat{\alpha}\hat{\gamma}} \wedge \hat{\boldsymbol{\omega}}_{\hat{\beta}}{}^{\hat{\gamma}} \wedge \star(\hat{\mathbf{e}}^{\hat{\alpha}} \wedge \hat{\mathbf{e}}^{\hat{\beta}})\,. \tag{5.13}$$

The component form of the above expression is

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \, \hat{e} \, \left(\hat{\omega}^{\hat{\alpha}}{}_{\hat{\alpha}\hat{\gamma}} \, \hat{\omega}^{\hat{\beta}}{}_{\hat{\beta}}{}^{\hat{\gamma}} + \hat{\omega}^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} \, \hat{\omega}^{\hat{\beta}\hat{\gamma}}{}_{\hat{\alpha}} \right), \tag{5.14}$$

where

$$\hat{\omega}_{\hat{\gamma}\hat{\alpha}\hat{\beta}} = -\hat{\Omega}_{\hat{\gamma}\hat{\alpha}\hat{\beta}} + \hat{\Omega}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} - \hat{\Omega}_{\hat{\beta}\hat{\gamma}\hat{\alpha}}. \tag{5.15}$$

The non-vanishing Ω's for the vielbein 5.3 and its inverse 5.4 are

$$\begin{aligned} \hat{\Omega}_{\alpha\beta\gamma} &= \frac{1}{2}(\hat{e}^{\mu}{}_{\alpha}\, \hat{e}^{\nu}{}_{\beta} - \hat{e}^{\mu}{}_{\beta}\, \hat{e}^{\nu}{}_{\alpha})\, \partial_\nu \hat{e}_{\mu\gamma} = \frac{1}{2} e^{-2\delta\Phi}(e^{\mu}{}_{\alpha}\, e^{\nu}{}_{\beta} - e^{\mu}{}_{\beta}\, e^{\nu}{}_{\alpha})\, \partial_\nu (e^{\delta\Phi} e_{\mu\gamma}) \\ &= e^{-\delta\Phi}\Omega_{\alpha\beta\gamma} + \frac{\delta}{2}\, e^{-\delta\Phi}(\eta_{\alpha\gamma} e^{\nu}{}_{\beta} - \eta_{\beta\gamma} e^{\nu}{}_{\alpha})\, \partial_\nu \Phi \end{aligned} \tag{5.16a}$$

$$\begin{aligned} \hat{\Omega}_{\alpha\beta 10} &= \frac{1}{2}(\hat{e}^{\mu}{}_{\alpha}\, \hat{e}^{\nu}{}_{\beta} - \hat{e}^{\mu}{}_{\beta}\, \hat{e}^{\nu}{}_{\alpha})\, \partial_\nu \hat{e}_{\mu 10} + \frac{1}{2}(\hat{e}^{10}{}_{\alpha}\, \hat{e}^{\nu}{}_{\beta} - \hat{e}^{10}{}_{\beta}\, \hat{e}^{\nu}{}_{\alpha})\, \partial_\nu e_{10\,10} \\ &= \frac{1}{2} e^{-2\delta\Phi}(e^{\mu}{}_{\alpha}\, e^{\nu}{}_{\beta} - e^{\mu}{}_{\beta}\, e^{\nu}{}_{\alpha})\, \partial_\nu (e^{\Phi} A_\mu) + \frac{1}{2} e^{-2\delta\Phi}(-A_\alpha e^{\nu}{}_{\beta} + A_\beta e^{\nu}{}_{\alpha})\, \partial_\nu e^{\Phi} \\ &= \frac{1}{2} e^{-(2\delta-1)\Phi}(\partial_\nu A_\mu - \partial_\mu A_\nu) = -\frac{1}{2} e^{-(2\delta-1)\Phi} F_{\mu\nu} \end{aligned} \tag{5.16b}$$

$$\hat{\Omega}_{\alpha 10\,10} = -\frac{1}{2}\hat{e}^{10}{}_{10}\, \hat{e}^{\nu}{}_{\alpha}\, \partial_\nu e_{10\,10} = -\frac{1}{2} e^{-(\delta+1)\Phi} e^{\nu}{}_{\alpha}\, \partial_\nu e^{\Phi} = e^{-\delta\Phi} e^{\nu}{}_{\alpha}\, \partial_\nu \Phi \tag{5.16c}$$

$$\hat{\Omega}_{10\alpha 10} = -\hat{\Omega}_{\alpha 10\,10}\,. \tag{5.16d}$$

Inserting these expressions in 5.15 we find the following spin connection components

$$\hat{\omega}_{\alpha\beta\gamma} = e^{-\delta\Phi}\omega_{\alpha\beta\gamma} + \delta e^{-\delta\Phi}(\eta_{\gamma\alpha}\, e^{\nu}{}_{\beta} - \eta_{\gamma\beta}\, e^{\nu}{}_{\alpha})\, \partial_\nu \Phi \tag{5.17a}$$

$$\hat{\omega}_{10\alpha\beta} = -\hat{\omega}_{\alpha\beta 10} = -\frac{1}{2} e^{-(2\delta-1)\Phi} F_{\mu\nu} \tag{5.17b}$$

$$\hat{\omega}_{10\,10\alpha} = -\hat{\omega}_{10\alpha 10} = e^{-\delta\Phi} e^{\nu}{}_{\alpha}\, \partial_\nu \Phi\,. \tag{5.17c}$$

Substituting the above spin connection components, the Einstein-Hilbert action 5.14 becomes

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2} \int d^{10}x\, d\rho\, \hat{e}\, \left(\hat{\omega}^{\alpha}{}_{\alpha\gamma}\, \hat{\omega}^{\beta}{}_{\beta}{}^{\gamma} + 2\, \hat{\omega}^{\alpha}{}_{\alpha\gamma}\, \hat{\omega}^{10}{}_{10}{}^{\gamma} + \hat{\omega}^{\alpha}{}_{\beta\gamma}\, \hat{\omega}^{\beta\gamma}{}_{\alpha} + \hat{\omega}^{10}{}_{\beta\gamma}\, \hat{\omega}^{\beta\gamma}{}_{10} \right) \tag{5.18}$$

where

$$\hat{\omega}^{\alpha}{}_{\alpha\gamma}\,\hat{\omega}^{\beta}{}_{\beta}{}^{\gamma} = e^{-2\delta\Phi}\,\omega^{\alpha}{}_{\alpha\gamma}\,\omega^{\beta}{}_{\beta}{}^{\gamma} + (9\delta)^2\,e^{-2\delta\Phi}\,\partial_\mu\Phi\partial^\mu\Phi \tag{5.19a}$$

$$2\,\hat{\omega}^{\alpha}{}_{\alpha\gamma}\,\hat{\omega}^{10}{}_{10}{}^{\gamma} = 18\delta\,e^{-2\delta\Phi}\,\partial_\mu\Phi\partial^\mu\Phi \tag{5.19b}$$

$$\hat{\omega}^{\alpha}{}_{\beta\gamma}\,\hat{\omega}^{\beta\gamma}{}_{\alpha} = e^{-2\delta\Phi}\,\omega^{\alpha}{}_{\beta\gamma}\,\omega^{\beta\gamma}{}_{\alpha} - 9\delta^2 e^{-2\delta\Phi}\,\partial_\mu\Phi\partial^\mu\Phi \tag{5.19c}$$

$$\hat{\omega}^{10}{}_{\beta\gamma}\,\hat{\omega}^{\beta\gamma}{}_{10} = -\frac{1}{4}\,e^{-(2\delta-1)2\Phi}F_{\mu\nu}F^{\mu\nu}\,. \tag{5.19d}$$

Using the above expressions and the relation $\hat{e} = e\,e^{(10\delta+1)\Phi}$, where e is the determinant of the vielbein in ten dimensions we find

$$I_{11}^{E-H} = \frac{1}{2\kappa_{11}^2}\int d^{10}x\,d\rho\,e\,e^{(10\delta+1)\Phi}e^{-2\delta\Phi}\Big[R(\omega) - \frac{1}{4}\,e^{-(\delta-1)2\Phi}F_{\mu\nu}F^{\mu\nu} + (72\delta^2+18\delta)\partial_\mu\Phi\partial^\mu\Phi\Big]\,. \tag{5.20}$$

The requirement that the Einstein-Hilbert action in ten dimensions have a canonical form i.e. $R(\omega)$ is not multiplied by an exponential of Φ, leads to $\delta = -\frac{1}{8}$. Using the above value of δ and integrating over the compactified dimension, we arrive at the following action in ten dimensions

$$I'_{10} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e\left(R(\omega) - \frac{1}{4}\,e^{\frac{9}{4}\Phi}F_{\mu\nu}F^{\mu\nu} - \frac{9}{8}\,\partial_\mu\Phi\partial^\mu\Phi\right), \tag{5.21}$$

where we have defined the gravitational coupling constant in ten dimensions

$$\kappa_{10}^2 \equiv \frac{\kappa_{11}^2}{2\pi R}\,. \tag{5.22}$$

Thus the Einstein-Hilbert Lagrangian in eleven dimensions reduces to an Einstein-Hilbert Lagrangian in ten dimensions and the kinetic terms for the scalar field $\Phi(x)$ and the vector gauge field $A_\mu(x)$ that appear in the decomposition of the metric.

The reduction of the 3-form gives rise to a 3-form and a 2-form in ten dimensions

$$\hat{A}_{\mu\nu\rho} = C_{\mu\nu\rho} \quad \text{and} \quad \hat{A}_{\mu\nu 10} = B_{\mu\nu}\,, \tag{5.23}$$

with corresponding field strengths

$$\hat{F}_{\mu\nu\kappa\lambda} = F_{\mu\nu\kappa\lambda} \quad \text{and} \quad \hat{F}_{\mu\nu\kappa 10} = G_{\mu\nu\kappa}\,. \tag{5.24}$$

The reduction of the kinetic term for the 3-form gauge field and the Chern-Simons term is performed more conveniently in the tangent space, where the metric is diagonal. We thus use the inverse vielbein to convert to tangent-space indices

$$\begin{aligned}\hat{F}_{\alpha\beta\gamma 10} &= \hat{e}^{\hat{\mu}}{}_{\alpha}\,\hat{e}^{\hat{\nu}}{}_{\beta}\,\hat{e}^{\hat{\kappa}}{}_{\gamma}\,\hat{e}^{10}{}_{10}\,\hat{F}_{\hat{\mu}\hat{\nu}\hat{\kappa}10} = e^{-\frac{5}{8}\Phi}G_{\alpha\beta\gamma}\\ \hat{F}_{\alpha\beta\gamma\delta} &= \hat{e}^{\hat{\mu}}{}_{\alpha}\,\hat{e}^{\hat{\nu}}{}_{\beta}\,\hat{e}^{\hat{\kappa}}{}_{\gamma}\,\hat{e}^{\hat{\lambda}}{}_{\delta}\,\hat{F}_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}} = e^{\frac{1}{2}\Phi}\left(F_{\alpha\beta\gamma\delta} - 4\,A_{[\delta}G_{\alpha\beta\gamma]}\right) = e^{\frac{1}{2}\Phi}\tilde{F}_{\alpha\beta\gamma\delta}\,.\end{aligned} \tag{5.25}$$

The kinetic term for $\hat{A}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ reduces to

$$\begin{aligned}\hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4} &= \hat{F}_{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3\hat{\alpha}_4}\hat{F}^{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3\hat{\alpha}_4}\\ &= e^{\Phi}\tilde{F}_{\alpha_1\alpha_2\alpha_3\alpha_4}\tilde{F}^{\alpha_1\alpha_2\alpha_3\alpha_4} + 4\,e^{-\frac{5}{4}\Phi}G_{\beta_1\beta_2\beta_3}G^{\beta_1\beta_2\beta_3}\end{aligned} \tag{5.26}$$

while the Chern-Simons term, after an integration by parts, reduces to

$$\begin{aligned}\hat{\epsilon}^{\hat{\mu}_1\ldots\hat{\mu}_{11}}A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}\hat{F}_{\hat{\mu}_4\ldots\hat{\mu}_7}\hat{F}_{\hat{\mu}_8\ldots\hat{\mu}_{11}} &= \hat{\epsilon}^{\hat{\alpha}_1\ldots\hat{\alpha}_{11}}A_{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3}\hat{F}_{\hat{\alpha}_4\ldots\hat{\alpha}_7}\hat{F}_{\hat{\alpha}_8\ldots\hat{\alpha}_{11}}\\ &= 3\,\epsilon^{\alpha_1\ldots\alpha_{10}}B_{\alpha_1\alpha_2}F_{\alpha_3\ldots\alpha_6}F_{\alpha_7\ldots\alpha_{10}}\\ &\qquad + 6\,\epsilon^{\alpha_1\ldots\alpha_{10}}C_{\alpha_1\alpha_2\alpha_3}F_{\alpha_4\ldots\alpha_7}G_{\alpha_8\alpha_9\alpha_{10}}\\ &\to 9\,\epsilon^{\alpha_1\ldots\alpha_{10}}B_{\alpha_1\alpha_2}F_{\alpha_3\ldots\alpha_6}F_{\alpha_7\ldots\alpha_{10}}\,.\end{aligned} \tag{5.27}$$

Aggregating all terms of the reduced bosonic sector of $d = 11$ supergravity, we obtain the following action in ten dimensions

$$\begin{aligned}I_{10}^{(b)} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\,e\,\Big[R(\omega) - \frac{1}{4}e^{\frac{9}{4}\Phi}F_{\mu\nu}F^{\mu\nu} - \frac{9}{8}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2\cdot 4!}\,e^{\frac{3}{4}\Phi}\tilde{F}_{\mu_1\mu_2\mu_3\mu_4}\tilde{F}^{\mu_1\mu_2\mu_3\mu_4}\\ &- \frac{1}{2\cdot 3!}\,e^{-\frac{3}{2}\Phi}G_{\nu_1\nu_2\nu_3}G^{\nu_1\nu_2\nu_3} + \frac{e^{-1}}{2\cdot 2!(4!)^2}\,\epsilon^{\mu_1\ldots\mu_{10}}B_{\mu_1\mu_2}F_{\mu_3\ldots\mu_6}F_{\mu_7\ldots\mu_{10}}\Big]\end{aligned} \tag{5.28}$$

The finishing touch would be to rescale the scalar field Φ by factor of $2/3$ so that the corresponding kinetic term has a conventional form. The above action is the bosonic sector of type IIA supegravity action [51, 52, 53].

In the fermionic sector, the gravitino decomposes to two gravitinos of opposite chirality and two spinors of opposite chirality, according to the $\mathsf{Spin}(9) \supset \mathsf{Spin}(8)$ representation decomposition [55]

$$\mathbf{128} \to \mathbf{56} + \mathbf{56} + \mathbf{8} + \mathbf{8}\,. \tag{5.29}$$

In terms of fields, the $10 + 1$ decomposition of the 11-dimensional gravitino gives rise to a

Majorana gravitino

$$\Psi_\mu = e^{\alpha}{}_{\mu}\, \hat{e}^{\hat{\mu}}{}_{\hat{\alpha}}\, \hat{\Psi}_{\hat{\mu}} = e^{\frac{1}{8}\Phi}(\hat{\Psi}_\mu - A_\mu \hat{\Psi}_{10}) \tag{5.30}$$

and a Majorana spinor

$$\psi = e^{\alpha}{}_{10}\, \hat{e}^{10}{}_{\hat{\alpha}}\, \Psi_{10} = e^{-\Phi}\hat{\Psi}_{10} \tag{5.31}$$

in ten dimensions, which can be further decomposed into two chiral eigenstates of opposite chirality. The latter compose the fermionic sector of type IIA supegravity.

The Lagrangian of type IIA supegravity in its full fermionic glory is presented in references [51], [52] and [53] .

5.2 Reduction of the membrane solution and the membrane

A natural next step would be to investigate what kind of solution emerges in ten dimensions, upon reducing a brane solution of the $d = 11$ supergravity equations. Thereupon, we consider the dimensional reduction of the membrane solution.

The membrane solution depends only on the 'transverse' coordinates and can be readily reduced to ten dimensions, by compactifying a 'world-volume' coordinate. This kind of reduction, where both the space-time dimension d and the brane dimension p are reduced, is called *diagonal dimensional reduction* [27]. Accordingly, we make a ten-one split of the space-time coordinates

$$\hat{x}^{\hat{M}} = (x^M, x^2)\,, \quad \hat{M} = 0, 1, 3, \ldots, 9 \tag{5.32}$$

and the following ansatz for the metric $\hat{g}_{\hat{M}\hat{N}}$ and the 3-form gauge field $\hat{A}_{\hat{M}\hat{N}\hat{P}}$ [38]

$$\hat{g}_{MN} = e^{-\Phi/6} g_{MN}, \quad \hat{g}_{22} = e^{4\Phi/3} \quad \text{and} \quad \hat{A}_{MN\,2} = B_{MN}. \tag{5.33}$$

All other components of $\hat{g}_{\hat{M}\hat{N}}$ and $\hat{A}_{\hat{M}\hat{N}\hat{P}}$ are set to zero. Upon insertion of the above decompositions into 4.18 and 4.19, the membrane solution reduces to

$$ds^2 = \left(1 + \frac{k_2}{r^6}\right)^{-3/4} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k_2}{r^6}\right)^{1/4} \delta_{mn} dx^m dx^n \tag{5.34}$$

and

$$B_{01} = \left(1 + \frac{k_2}{r^6}\right)^{-1}, \quad e^{\Phi} = \left(1 + \frac{k_2}{r^6}\right)^{-1/2} \tag{5.35}$$

The above equations define the *elementary BPS string* solution of $d = 10$ supergravity described in [37].

In addition to the reduction of the membrane solution, a *double dimensional reduction* of the supermembrane leads to the superstring in ten dimensions. The double dimensional reduction is performed by reducing a supermembrane coupled to a $d = 11$ supergravity background from 11 to 10-dimensional space-time and simultaneously from 3 to 2-dimensional world-volume. The result is a superstring coupled to a type IIA supergravity background [54]. We will demonstrate the aforementioned reduction for the bosonic sector of the supermembrane.

Let us start with the membrane action 4.26 in eleven dimensions and make a two-one split of the world-volume coordinates

$$\xi^I = (\xi^i, \xi^2)\,, \qquad I = 0, 1, 2 \qquad i = 0, 1 \tag{5.36}$$

and a ten-one split of the space-time coordinates

$$X^M = (X^\mu, X^{10})\,, \qquad \mu = 0, \ldots, 9\,. \tag{5.37}$$

The crucial step is the identification of the compactified world-volume coordinate with the compactified space-time coordinate, through the partial gauge choice

$$\xi^2 = X^{10} \equiv \rho\,. \tag{5.38}$$

Effectively, the membrane 'wraps around' the compactified dimension so that the membrane world-volume and the space-time are compactified over the same circle. The $10 + 1$ ansatz for the 11-dimensional space-time metric is [54]

$$\hat{g}_{MN} = e^{-2\Phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} A_\mu A_\nu & e^{2\Phi} A_\nu \\ e^{2\Phi} A_\mu & e^{2\Phi} \end{pmatrix}, \tag{5.39}$$

while the 3-form gauge field and the 4-form field strength decompose as in 5.23 and 5.24 respectively. We discard the massive modes of the space-time and world-volume fields by imposing

$$\partial_\rho \hat{g}_{MN} = 0 = \partial_\rho \hat{A}_{MNP} \quad \text{and} \quad \partial_\rho X^\mu = 0\,. \tag{5.40}$$

Using 4.28 we find the induced metric on the world-volume of the membrane to be

$$\hat{\gamma}_{IJ} = e^{-2\Phi/3} \begin{pmatrix} \gamma_{ij} + e^{2\Phi} A_i A_j & e^{2\Phi} A_i \\ e^{2\Phi} A_j & e^{2\Phi} \end{pmatrix} , \tag{5.41}$$

where

$$\gamma_{ij} \equiv \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} \quad \text{and} \quad A_i \equiv \partial_i X^\mu A_\mu \,. \tag{5.42}$$

Inserting the $10+1$ decompositions 5.39, 5.41 and 5.24 in the equation of motion 4.27 of the membrane gives for $M = \mu$

$$\partial_i \left(\sqrt{-\gamma}\, \gamma^{ij} \partial_i X^\nu g_{\mu\nu} \right) + \frac{1}{2} \sqrt{-\gamma}\, \gamma^{ij} \partial_i X^\nu \partial_j X^\rho \partial_\mu g_{\nu\rho} + \frac{1}{2} \epsilon^{ij} \partial_i X^\nu \partial_j X^\rho G_{\mu\nu\rho} = 0 \,, \tag{5.43}$$

while for $M = \rho$ the resulting equation is an identity, as it must be for consistency.

The above equation is the equation of motion of a bosonic string in ten space-time dimensions coupled to the 2-form gauge field $B_{\mu\nu}$ of type IIA supergravity. Note that the gauge fields $C_{\mu\nu\rho}$ and A_μ and the scalar field Φ appear to have decoupled. What actually happens is that they survive in the fermionic sector [54]. Equation 5.43 can be derived from the action

$$S_1 = T_1 \int d^2\xi \left(-\frac{1}{2} \sqrt{-\gamma}\, \gamma^{ij}\, \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + \frac{1}{2} \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu B_{\mu\nu} \right) , \tag{5.44}$$

which is no other than the bosonic sector of the superstring action coupled to a type IIA supergravity background [32].

We have worked in the limit that the radius R of the compactified dimension tends to zero and only the massless modes survive. At the same time, the membrane tension T_2 tends to infinity, so that the string tension $T_1 = 2\pi R T_2$ remains finite.

Appendix A

The vielbein formalism

Spinor fields in d-dimensional Minkowski space-time appear as representations of the double cover of the Lorentz group, $\mathsf{Spin}(1, \mathsf{d}-1)$. The standard formalism of General Relativity does not allow the introduction of fermions, since the group of general coordinate transformations $\mathsf{GL}(\mathsf{d}, \mathbb{R})$ does not admit spinorial representations. To circumvent this problem, one formulates General Relativity in terms of *vielbeins* [56].

Equivalence principle states that the tangent vector space T_pM at a point p of a d-dimensional space-time manifold M with metric g, admits an orthonormal basis $\{\hat{e}_\alpha\}$ of d vectors

$$g(\hat{e}_\alpha, \hat{e}_\beta) = \eta_{\alpha\beta}\,, \tag{A.1}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric. Early letters of the Greek alphabet $(\alpha, \beta, \gamma, \dots)$ are used for tangent-space indices and late letters of the Greek alphabet $(\mu, \nu, \rho, \dots)$ are used for base-space indices. The orthonormal basis $\{\hat{e}_\alpha\}$ is related to the coordinate basis $\{\hat{e}_\mu\}$ as

$$\hat{e}_\alpha = e^\mu{}_\alpha\, \hat{e}_\mu\,, \qquad e^\mu{}_\alpha \in \mathsf{GL}(\mathsf{d}, \mathbb{R}). \tag{A.2}$$

The inverse $e^\alpha{}_\mu$ of the matrix $e^\mu{}_\alpha$ is the vielbein, which defines the one-form $\mathbf{e}^\alpha = e^\alpha{}_\mu dx^\mu$. The vielbein is determined up to a local Lorentz transformation

$$e^\alpha{}_\mu \to \Lambda^\alpha{}_\beta\, e^\beta{}_\mu\,, \qquad \Lambda^\alpha{}_\beta \in \mathsf{SO}(1, \mathsf{d}-1)\,, \tag{A.3}$$

that reduces its independent components from d^2 to $\frac{1}{2}d(d+1)$, the number of independent components of the metric.

The vielbeins and their inverses allow the transition between the tangent vector space and the base manifold; the metric $g_{\mu\nu}$ of the base manifold is related to the Minkowski metric $\eta_{\alpha\beta}$ of the tangent space as

$$g_{\mu\nu} = e^{\alpha}{}_{\mu}\, e^{\beta}{}_{\nu}\, \eta_{\alpha\beta} \quad \text{and} \quad \eta_{\alpha\beta} = e^{\mu}{}_{\alpha}\, e^{\nu}{}_{\beta}\, g_{\mu\nu}\,. \tag{A.4}$$

The covariant derivative for the local Lorentz transformations is

$$D_\mu \equiv \partial_\mu + \frac{1}{2}\omega_\mu{}^{\alpha\beta} M_{\alpha\beta}\,, \tag{A.5}$$

where $M_{\alpha\beta} = -M_{\beta\alpha}$ are the generators of the Lorentz group and $\omega_\mu{}^{\alpha\beta}$ is the *spin connection*, acting as the gauge field for the local $\mathsf{Spin}(1, \mathsf{d}-1)$ group. The spin connection can be expressed in terms of the vielbein upon imposing the torsion free condition $T^a_{\mu\nu} \equiv D_{[\mu} e^{\alpha}{}_{\nu]} = 0$. The result is

$$\omega_\mu{}^{\alpha\beta} = e_{\gamma\mu}\left(\Omega^{\alpha\beta\gamma} - \Omega^{\beta\gamma\alpha} - \Omega^{\gamma\alpha\beta}\right) \tag{A.6}$$

where

$$\Omega_{\alpha\beta\gamma} = \frac{1}{2}(e^{\mu}{}_{\alpha}\, e^{\nu}{}_{\beta} - e^{\mu}{}_{\beta}\, e^{\nu}{}_{\alpha})\partial_\nu e_{\mu\gamma}. \tag{A.7}$$

Spinors are introduced as representations of the local $\mathsf{Spin}(1, \mathsf{d}-1)$ group. The covariant derivative of a spinor Ψ is

$$D_\mu\Psi = \partial_\mu\Psi + \frac{1}{2}\omega_\mu{}^{\alpha\beta} S_{\alpha\beta}\Psi\,, \tag{A.8}$$

where $S_{\alpha\beta} = \frac{1}{4}[\Gamma_\alpha, \Gamma_\beta]$ are the generators of the Lorentz group in the spinorial representation.

In analogy with a Yang-Mills theory, the field strength (or curvature form) of the spin connection is defined as

$$R_{\mu\nu}{}^{\alpha\beta} = \partial_\mu\omega_\nu{}^{\alpha\beta} - \partial_\nu\omega_\mu{}^{\alpha\beta} + \omega_\mu{}^{\alpha\gamma}\,\omega_{\nu\gamma}{}^{\beta} - \omega_\nu{}^{\alpha\gamma}\,\omega_{\mu_\gamma}{}^{\beta}\,. \tag{A.9}$$

The curvature form is related to the Riemann curvature tensor $R^{\mu}{}_{\nu\kappa\lambda}$ via the expression

$$R^{\mu}{}_{\nu\kappa\lambda} = e^{\mu}{}_{\alpha}\, e_{\beta\nu}\, R_{\kappa\lambda}{}^{\alpha\beta}. \tag{A.10}$$

Appendix B

Clifford algebra and spinors in eleven dimensions

The Clifford algebra in eleven space-time dimensions is generated by gamma matrices Γ_μ which satisfy the relation

$$\{\Gamma_\mu, \Gamma_\nu\} \equiv \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu}\,, \qquad \mu, \nu = 0, \ldots, 10 \tag{B.1}$$

where η is the metric of flat space-time with signature $(-, +, \ldots, +)$. There are two inequivalent irreducible representations of the Clifford algebra and both have dimension $2^{\frac{11-1}{2}} = 32$. They differ according to whether the product $\Gamma^0 \Gamma^1 \ldots \Gamma^{10}$ equals $\mathbf{1}$ or $-\mathbf{1}$.

The complex conjugate Γ_μ^* and the transpose Γ_μ^t of the original representation Γ_μ, form equivalent representations of the Clifford algebra B.1. Consequently, there exists a matrix B such that

$$\Gamma_\mu^* = B \Gamma_\mu B^{-1} \tag{B.2}$$

and a matrix C, called the charge conjugation matrix, such that

$$\Gamma_\mu^t = -C \Gamma_\mu C^{-1}. \tag{B.3}$$

In eleven dimensions B satisfies $B^* B = \mathbf{1}$ and can be set equal to the identity $\mathbf{1}$, while C satisfies $C^t = -C$. Accordingly, gamma matrices obey the reality condition $\Gamma_\mu^* = \Gamma_\mu$. Furthermore, the representation Γ_μ can be chosen to be unitary i.e. $\Gamma_\mu \Gamma_\mu^\dagger = \mathbf{1}$. Since $\Gamma_\mu \Gamma_\mu = \eta_{\mu\mu}$ we conclude that

$$\Gamma_\mu^\dagger = \eta_{\mu\nu} \Gamma_\nu \tag{B.4}$$

or equivalently

$$\Gamma_\mu^\dagger = \Gamma_0 \Gamma_\mu \Gamma_0 \, . \tag{B.5}$$

Combining B.2, B.3, B.5 and $B = \mathbf{1}$ yields the expression $C = \Gamma_0$.

The relation $B^* B = \mathbf{1}$ allows the existence of Majorana spinors, defined as spinors ψ satisfying

$$\psi^* = B\psi. \tag{B.6}$$

Majorana spinors are the minimal spinors in eleven dimensions and consist of thirty-two real components.

A detailed discussion on Clifford algebras and spinors in various space-time dimensions can be found in references [14] and [28].

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