Who Should Sell Stocks?

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Imperial-ETH Workshop on Mathematical Finance 2015
Merton’s Problem (1969)

- Frictionless market consisting of one safe and one risky asset
- Constant investment opportunities and CRRA for the investor
- Maximize the expected utility of final wealth
- **Solution**: risky weight $\pi_t \equiv \pi_*$
Merton’s Problem with Proportional Transaction Costs


- No trading, if the risky weight is inside a certain no-trade region
- Minimal trading (of local-time type), if the boundaries of the no-trade region are breached
Merton’s Problem with Transaction Costs and Continuous Dividends
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Merton’s Problem with $\varepsilon = 1\%$
Merton’s Problem with Transaction Costs and Continuous Dividends

Merton’s Problem and with $\varepsilon = 1\%$ and $\delta = 3\%$
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Motivation

- Buy-and-hold is only optimal for very particular preferences.
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- Jang 2007: numerical approach, but no new effect
Merton’s problem with prop. transaction costs and continuous dividends: dynamic Buy-and-Hold can be optimal for a range of realistic parameters
This paper

- Merton’s problem with prop. transaction costs and continuous dividends: dynamic Buy-and-Hold can be optimal for a range of realistic parameters
- Dividends are relevant for the portfolio choice problem in contrast to capital structure (M&M theorem)
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More complicated model might lead to simpler optimal solutions
This paper

- Merton’s problem with prop. transaction costs and continuous dividends: dynamic Buy-and-Hold can be optimal for a range of realistic parameters
- Dividends are relevant for the portfolio choice problem in contrast to capital structure (M&M theorem)
- More complicated model might lead to simpler optimal solutions
- Closed form optimal strategies even with capital gains tax
Standing Assumptions:

- Black-Scholes dynamics with continuous dividends:
  \[ \frac{dS_t}{S_t} = (r + \mu - \delta)dt + \sigma dW_t \]

- Proportional Transaction Costs: buy at the ask price \((1 + \varepsilon)S\), sell at the bid price \((1 - \varepsilon)S\)

- Constant Relative Risk Aversion \(0 < \gamma \neq 1\)

- Infinite planning horizon

- Frictionless solution: \(0 < \pi_* = \frac{\mu}{\gamma \sigma^2} < 1\), i.e., no short or levered positions
Goal: maximize the equivalent safe rate ESR among all admissible strategies:

$$\max \left( \liminf_{T \to \infty} \frac{1}{T} \log \mathbb{E} \left[ (\Xi_T)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right)$$

- \(\Xi_t\) = liquidation value at time \(t\)
- admissible "=" self financing and \(\Xi_t \geq 0\)
Main Results: Parameter assumption

Set

\[ \pi^\dagger_\pm(\lambda) = \frac{\mu \pm \epsilon \delta/(1 \mp \epsilon) \pm \sqrt{\lambda^2 \pm 2\mu \epsilon \delta/(1 \mp \epsilon) + (\epsilon \delta/(1 \mp \epsilon))^2}}{\gamma \sigma^2} \]

\[ \pi_-(\lambda) = \pi^\dagger_-(\lambda), \quad \pi_+(\lambda) = \min\left(\pi^\dagger_+, 1\right). \]

Suppose one of the following condition is satisfied:

(a) there exists \( \lambda > 0 \) such that \( \pi_+(\lambda) < 1 \) and the solution \( w(\cdot, \lambda) \) of terminal value problem also satisfies a certain initial condition.

(b) there exists \( \lambda > 0 \) such that \( \pi_+(\lambda) = 1 \) and the solution \( w(\cdot, \lambda) \) of a Riccati ODE with a limit condition at infinity also satisfies a certain initial condition.
Main Results: Optimal Policy

Theorem

In the presence of proportional transaction costs \( \varepsilon > 0 \) and a continuous yield \( \delta > 0 \) an investor trades to maximizes the equivalent safe rate. Then, under the previous assumption we have:

- It is optimal to keep the risky weight within the buying and selling boundaries \([\pi_-, \pi_+]\)
- The optimal equivalent safe rate \( \beta = r + (\mu^2 - \lambda^2)/2\gamma\sigma^2 \)
- In case of \( \pi_+ < 1 \) it holds

\[
\pi_{\pm} = \pi_* \pm \left( \frac{3}{2\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{1/3} \varepsilon^{1/3} \\
+ \frac{\delta}{\gamma\sigma^2} \left( \frac{2\gamma\pi_*}{3 (1 - \pi_*)^2} \right)^{1/3} \varepsilon^{2/3} + \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0
\]
Figure: The no-trade region (vertical axis) plotted against the dividend yield $\delta$ (horizontal axis) for $\gamma = 3.45$ ($\pi^* = 90.6\%$), $\mu = 8\%$, $\sigma = 16\%$ and $\varepsilon = 1\%$. 
Figure: The never-sell region (shaded) for pairs of dividend yield $\delta$ (horizontal axis) and frictionless portfolio weight $\pi_*$ (vertical axis). Parameters are $\mu = 8\%$, $\sigma = 16\%$ and $\varepsilon = 1\%$. 
Robustness

<table>
<thead>
<tr>
<th>π*</th>
<th>optimal</th>
<th>never sell</th>
<th>buy &amp; hold</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td>1.67%</td>
<td>2.00%</td>
<td>4.67%</td>
</tr>
<tr>
<td>70%</td>
<td>1.58%</td>
<td>1.58%</td>
<td>4.21%</td>
</tr>
<tr>
<td>90%</td>
<td>1.52%</td>
<td>1.52%</td>
<td>3.70%</td>
</tr>
</tbody>
</table>

**Table:** Relative equivalent safe rate loss of the optimal ([π−, π+]), never sell ([π−, 1]) and buy-and-hold ([0, 1]). These numbers are computed using Monte Carlo simulation with \( T = 20 \), time step \( dt = 1/250 \) and sample size \( N = 2 \cdot 10^7 \), \( \mu = 8\% \), \( \sigma = 16\% \), \( r = 1\% \), \( \delta = 2\% \), and \( \varepsilon = 1\% \).
Robustness with respect to Taxes

- **Dividend Tax**: suppose the effective dividend rate $= \delta(1 - \tau)$ with $0 < \tau < 1$ and the expected, ex-dividend return remains $\mu - \delta$
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- This model is equivalent to a model without dividend tax but with a dividend yield \( \tilde{\delta} = \delta(1 - \tau) \) and expected total return \( \tilde{\mu} = \mu - \delta \tau \).
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- **Capital Gains Tax**: Sales of the risky asset induces a tax payment or credit of $\alpha (S_t - B_t)$ with $0 < \alpha < 1$ ($B$ is the cost basis process/reference value)
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Taxes

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<tr>
<th>$\pi_*$</th>
<th>$[\pi_-, \pi_+]_{\text{ave}}$</th>
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<th>never sell</th>
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<tbody>
<tr>
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<td>2.41%</td>
<td>2.41%</td>
<td>2.07%</td>
<td>4.48%</td>
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<tr>
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<td>1.91%</td>
<td>1.91%</td>
<td>1.64%</td>
<td>3.55%</td>
</tr>
<tr>
<td>90%</td>
<td>1.36%</td>
<td>1.36%</td>
<td>1.36%</td>
<td>2.94%</td>
</tr>
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**Table:** Relative equivalent safe rate loss of the capital gains tax adjusted optimal ($[\pi_-, \pi_+]$), never sell ($[\pi_-, 1]$) and buy-and-hold ($[0, 1]$). These numbers are computed using Monte Carlo simulation with $T = 20$, time step $dt = 1/250$ and sample size $N = 2 \cdot 10^7$, $\mu = 8\%$, $\sigma = 16\%$, $\alpha = 20\%$, $\tau = 20\%$, $r = 1\%$, $\delta = 2\%$ and $\varepsilon = 1\%$. 
Objective function cf. Janecek and Shreve (2004), Shreve and Soner (1994)

$$\max \left( \frac{1}{1 - \gamma} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} C_t^{1-\gamma} dt \right] \right)$$

For $\varepsilon = 0$ we have

$$\frac{C^*_t}{X_t + Y_t} = \frac{\rho}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) \left( r + \frac{\mu^2}{2\gamma \sigma^2} \right)$$

This consumption policy is approximately optimal even with small proportional transaction costs (Kallsen and Muhle-Karbe 2013)
Consumption

<table>
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<th>( \pi_* )</th>
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<td>1.05%</td>
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Suggestions and Limitations

- Retirement planning: investors with moderate risk aversions should avoid selling.
- After the retirement: gradually liquidate stocks to finance the required consumption or invest in high dividend funds.
- Dynamic Buy-and-Hold might be suboptimal for:
  - small transaction costs
  - low dividend yields
  - large risk aversions
  - high consumption rates
Heuristic Derivation

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- We use a "power" transformation (cf. Jang (2007)) of the HJB equation $\leadsto$ Whittaker equation (explicit solutions in terms of the Whittaker functions)
- The boundary conditions yield the characterization of the gap parameter $\lambda$
Construction of Shadow Market \((S^0, \tilde{S})\)

**Shadow Price Process** \(\tilde{S}\):

- Lies within the bid-ask spread \([ (1 - \varepsilon)S, (1 + \varepsilon)S ] \) a.s.
- Existence of a long-run optimal strategy, i.e.,
  - Finite variation strategy
  - Self-financing strategy and solvent w.r.t. \(\tilde{S}\)
  - Maximizes the equivalent safe rate w.r.t. \(\tilde{S}\)
  - Same dividend payments \(\tilde{\delta} \tilde{S} = \delta S\)
  - Entails buying only when \(\tilde{S}_t = (1 + \varepsilon)S_t\)
  - Entails selling only when \(\tilde{S}_t = (1 - \varepsilon)S_t\)
Verification

- Optimality of the candidate strategy in shadow market (cf. Guasoni and Robertson 2012)
  - (super-) Martingale measure $\Rightarrow$ upper bound of the finite horizon ESR
  - Candidate strategy $\Rightarrow$ lower bound of the finite horizon ESR
  - Upper bound $=$ lower bound as $T \to \infty$

- Optimality of the candidate strategy in original market
  - Property of the shadow market
Thank You!