Functional limit theorems for generalized variations of the fractional Brownian sheet

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Introduction

- We are interested in the asymptotic behavior of non-linear functionals of high-frequency observations of strongly autocorrelated Gaussian random fields.
- By random fields, we mean stochastic processes with multiple parameters.
- In particular, we study the “phase transition” from Gaussian limits (central limit theorems) to non-Gaussian limits.
- We are also interested in the qualitative differences between the Gaussian and non-Gaussian limits (aside from Gaussianity/non-Gaussianity).
Review of the one-parameter case

Fractional Brownian sheet

Functional limit theorems
Fractional Brownian motion

- Let $Z_H = \{Z_H(t) : t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$.

- That is, $Z_H$ is a centered Gaussian process with covariance

\[
E[Z_H(t)Z_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}.
\]

- Specifically, we represent $Z_H$ as $Z_H(t) = \int_{\mathbb{R}} G_H(t, u)dW(u)$, $t \in \mathbb{R}$, where $W$ is a standard Brownian motion and

\[
G_H(t, u) := C(H)((t - u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}), \quad t, u \in \mathbb{R},
\]

is the Mandelbrot–Van Ness kernel. Above, $x_+ := \max(x, 0)$ and $C(H) > 0$ is a constant that depends on $H$. 
Hermite polynomials

• Let $P_k$, $k = 1, 2, \ldots$, be the Hermite polynomials given by

$$P_1(x) = x,$$
$$P_2(x) = x^2 - 1,$$
$$P_3(x) = x^3 - 3x,$$
$$P_4(x) = x^4 - 6x^2 + 3,$$
$$\vdots$$

• They are orthogonal polynomials with respect to the Gaussian measure on $\mathbb{R}$. 
Hermite variations

• Let \( k \geq 2 \) and \( n \in \mathbb{N} \).

• The \( k \)-th Hermite variation of the fBm \( Z_H \) on the grid \( \{1/n : 0, 1, 2, \ldots, n\} \) is defined as

\[
V_k^{(n)}(t) := \sum_{k=1}^{\lfloor nt \rfloor} P_k \left( n^H \left( Z_H \left( \frac{i}{n} \right) - Z_H \left( \frac{i-1}{n} \right) \right) \right), \quad t \in [0, 1].
\]

• The realizations of the process \( V_k^{(n)} \) belong to the Skorohod space \( D([0, 1]) \).
Functional central limit theorem

- It follows from the classical results of Breuer and Major (1983) and Taqqu (1977) that if $H \in \left(0, 1 - \frac{1}{2k}\right)$, then

\[
\left(Z_H, n^{-1/2} V_k^{(n)}\right) \xrightarrow{d_{n \to \infty}} \left(Z_H, C'(H, k)B\right) \quad \text{in } D([0, 1])^2,
\]

where $C'(H, k) > 0$ is a constant and $B$ is a standard Brownian motion, independent of $Z_H$.

- Moreover, in the critical case $H = 1 - \frac{1}{2k}$ it follows that

\[
\left(Z_H, (n \log n)^{-1/2} V_k^{(n)}\right) \xrightarrow{d_{n \to \infty}} \left(Z_H, C'(1 - \frac{1}{2k}, k)B\right) \quad \text{in } D([0, 1])^2.
\]
Non-central limit theorem

- When $H \in (1 - \frac{1}{2k}, 1)$, the Hermite variations have a limit, under suitable scaling, but the limit is non-Gaussian.
- More specifically, the result of Dobrushin and Major (1979) implies a non-central limit theorem: for any $t \in [0, 1]$,

$$n^{-(1-k(1-H))} V_k^{(n)}(t) \xrightarrow{L^2} C'(H, k) Y(t),$$

where \{ $Y(t) : t \in [0, 1]$ \} is a $k$-th order Hermite process with Hurst parameter $1 - k(1 - H) \in \left( \frac{1}{2}, 1 \right)$.
- The Hermite process can be represented as a $k$-fold multiple Wiener integral with respect to Brownian motion.
- The second-order Hermite process is also known as the Rosenblatt process and its marginals are infinitely divisible.
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Fractional Brownian sheet

- The fractional Brownian sheet (fBs), introduced by Ayache, Leger, and Pontier (2002), is an extension of the fBm to a multiparameter setting.

- It is defined by taking the “tensor product” of the correlation structures of multiple fBms with different Hurst parameters.

- More concretely, a $d$-parameter fBs with Hurst parameter $\mathbf{H} = (H_1, \ldots, H_d) \in (0, 1)^d$ is a centered Gaussian process $\{Z_{\mathbf{H}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^d\}$ with covariance

\[
E[Z_{\mathbf{H}}(\mathbf{t})Z_{\mathbf{H}}(\mathbf{u})] = \prod_{\nu=1}^{d} \frac{1}{2}(|t_{\nu}|^{2H_{\nu}} + |u_{\nu}|^{2H_{\nu}} - |t_{\nu} - u_{\nu}|^{2H_{\nu}}),
\]

for $\mathbf{t} = (t_1, \ldots, t_d), \mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$. 

The fBs is self-similar and has stationary increments (in the multiparameter sense). Moreover, it admits a continuous modification.

But the smoothness properties of the realizations depend on the direction (anisotropy).

Obviously, in the case $d = 1$ we recover the fBm.

We will use the representation

$$Z_H(t) = \int_{\mathbb{R}^d} \prod_{\nu=1}^{d} G_{H,\nu}(t, s_{\nu}) \mathcal{W}(ds_1, \ldots, ds_d), \quad t \in \mathbb{R}^d,$$

where $G_H$ is the Mandelbrot–Van Ness kernel and $\{\mathcal{W}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a white noise on $\mathbb{R}^d$. 
Simulation of the two-parameter fBIs

H = (0.5,0.5)
Simulation of the two-parameter fBzs

$H = (0.1, 0.1)$
Simulation of the two-parameter fBWs

$H = (0.9, 0.9)$
Simulation of the two-parameter fBSs

$H = (0.1, 0.9)$
Simulation of the two-parameter fBs

$H = (0.9, 0.5)$
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Functional limit theorems
Overview

- We want to extend the limit theorems for the fBm (as seen in the beginning) to the multiparameter setting with the fBs.
- Instead of Hermite variations, we consider more general functionals, generalized variations, where the Hermite polynomial is replaced with a more general function.
- When is the limit Gaussian?
  - Consider, e.g., the non-obvious case where $H_1$ is in the Gaussian regime and $H_2$ is in the non-Gaussian one.
- We extend the results of Réveillac, Stauch, and Tudor (2012).
Increments in the multiparameter setting

- Consider a function $h : \mathbb{R}^d \rightarrow \mathbb{R}$.
- The increment of $h$ over a hyper-rectangle

$$[a, b) = [a_1, b_1) \times \cdots \times [a_d, b_d),$$

where $a, b \in \mathbb{R}^d$, is given by

$$h([a, b)) : = \sum_{i\in\{0,1\}^d} (-1)^{d-\sum_{\nu=1}^d i_\nu} h((1 - i)a + ib).$$

(Above, vectors are multiplied component-wise.)

- This definition can be recovered by differencing iteratively with respect to each of the arguments of the function $h$. 
Generalized variations

- Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( \mathbb{E}[f(\xi)^2] < \infty \) and \( \mathbb{E}[f(\xi)] = 0 \) with \( \xi \sim N(0, 1) \).
- Moreover, for \( i \in \mathbb{N}^d \) and \( n \in \mathbb{N} \), denote
  \[
  \Box_i(n) := \left[ \frac{i_1 - 1}{n}, \frac{i_1}{n} \right) \times \cdots \times \left[ \frac{i_d - 1}{n}, \frac{i_d}{n} \right).
  \]
- We study the generalized variations of the fBs \( Z_{H_i} \),
  \[
  U_f^{(n)}(t) := \sum_{1 \leq i \leq \lfloor nt \rfloor} f \left( n\sum_{\nu=1}^{d} H_{\nu} Z_{H_i} \left( \Box_i(n) \right) \right), \quad t \in [0, 1]^d,
  \]
  for \( n \in \mathbb{N} \). (Above, all operations and relations involving vectors are understood component-wise.)
- The realizations of \( U_f^{(n)} \) belong to the multiparameter Skorohod space \( D([0, 1]^d) \).
Hermite expansion

- The function $f$ can be expanded in $L^2(\mathbb{R}, \gamma)$, where $\gamma$ stands for the $N(0, 1)$ distribution, using Hermite polynomials as

$$ f(x) = \sum_{k=k}^{\infty} a_k P_k(x), $$

where $a_k, a_{k+1}, \ldots$ are such that $a_k \neq 0$ and $\sum_{k=k}^{\infty} k! a_k^2 < \infty$.

- The index $k \in \mathbb{N}$ is known as the Hermite rank of $f$.

Standing assumption

The coefficients $a_k, a_{k+1}, \ldots$ satisfy $\sum_{k=k}^{\infty} 3^{\frac{k}{2}} \sqrt{k!} |a_k| < \infty$. 
Rescaling

- The generalized variations need to be rescaled, in a way that depends on $H$ and $k$, to ensure convergence.
- To this end, we define for any $\nu = 1, \ldots, d$ and $n \in \mathbb{N}$,

$$
\tau_{\nu}^{(n)} := \begin{cases} 
  n^{-\frac{1}{2}}, & H_{\nu} \in (0, 1 - \frac{1}{2k}), \\
  (n \log n)^{-\frac{1}{2}}, & H_{\nu} = 1 - \frac{1}{2k}, \\
  n^{-(1-k(1-H_{\nu}))}, & H_{\nu} \in (1 - \frac{1}{2k}, 1).
\end{cases}
$$

- We define the rescaled variations by

$$
\overline{U}_{f}^{(n)} := \left( \prod_{\nu=1}^{d} \tau_{\nu}^{(n)} \right) U_{f}^{(n)}, \quad n \in \mathbb{N}.
$$
Theorem

Suppose that $\mathbf{H} \in (0, 1)^d \setminus (1 - \frac{1}{2k}, 1)^d$. Then,

$$
\left( Z_H, \bar{U}_f(n) \right) \overset{d}{\underset{n \to \infty}{\longrightarrow}} \left( Z_H, C''(H, f) \tilde{Z}_{\tilde{H}} \right) \quad \text{in } D([0, 1]^d)^2,
$$

where $C''(H, f) > 0$ is a constant and $\tilde{Z}_{\tilde{H}}$ is a new fBS, independent of $Z_H$, with Hurst parameter $\tilde{H} \in \left[ \frac{1}{2}, 1 \right)$ given by

$$
\tilde{H}_\nu = \begin{cases} 
\frac{1}{2}, & H_\nu \in \left( 0, 1 - \frac{1}{2k} \right], \\
1 - k(1 - H_\nu), & H_\nu \in \left( 1 - \frac{1}{2k}, 1 \right),
\end{cases}
$$

for any $\nu = 1, \ldots, d$. 

Functional central limit theorem
Hermite sheet

- To describe the limit in the case $\mathbf{H} \in (1 - \frac{1}{2k}, 1)^d$, we need the so-called Hermite sheet.

- A $k$-th order $d$-parameter Hermite sheet with Hurst parameter $\mathbf{H} \in (\frac{1}{2}, 1)^d$ is a process $\{Y_{\mathbf{H}}(t) : t \in [0, \infty)^d\}$ given by

$$Y_{\mathbf{H}}(t) := \int_{(\mathbb{R}^d)^k} J^{(k)}_{\mathbf{H}}(t, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) \mathcal{W}(d\mathbf{u}^{(1)}) \cdots \mathcal{W}(d\mathbf{u}^{(k)}),$$

where $\mathcal{W}$ is the white noise on $\mathbb{R}^d$ and

$$J^{(k)}_{\mathbf{H}}(t, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})$$

$$:= C'''(\mathbf{H}, k) \int_{[0, t]} \prod_{\kappa=1}^k \prod_{\nu=1}^d (y_{\nu} - u^{(\kappa)}_{\nu})^{-\frac{1}{2} - \frac{1-H_{\nu}}{k}} dy$$

for $t \in [0, \infty)^d$ and $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)} \in \mathbb{R}^d$, with constant $C'''(\mathbf{H}, k) > 0$. 
Hermite sheet (continued)

- This representation of the Hermite sheet is due to Clarke de la Cerda and Tudor (2014).
- The Hermite sheet has the same correlation structure (and self-similarity properties) as the fBs, but it is non-Gaussian whenever $k \geq 2$.
- In the case $k = 1$ it coincides with the fBs.
- In the case $d = 1$ it reduces to the Hermite process.
Functional non-central limit theorem

**Theorem**

Suppose that $H \in (1 - \frac{1}{2k}, 1)^d$. Then,

$$U^{(n)}_f \xrightarrow{p} C^{''''}(H, f) Y_{\tilde{H}} \quad \text{in } D([0,1]^d),$$

where $C^{''''}(H, f) > 0$ is a constant and $Y_{\tilde{H}}$ is a $k$-th order Hermite sheet with Hurst parameter $\tilde{H} \in (\frac{1}{2}, 1)^d$ given by

$$\tilde{H}_\nu = 1 - k(1 - H_\nu) \quad \text{for any } \nu = 1, \ldots, d.$$

**Remark**

The Hermite sheet $Y_{\tilde{H}}$ is driven by the same white noise $\mathcal{W}$ as the original fBss $Z_H$. 
References


