The price impact of trades

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Limit order books

Some important aspects of microstructure

Mesoscopic microstructure

Empirical analysis on the BitCoin/USD exchange market

Applications and further aspects
The limit order book

Virtually all modern financial markets are based on *limit order books* which record buy or sell orders with specified quantities and prices. *Transactions* take place when limit orders cross: Market order (MO)
The limit order book

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Example: Sell limit order crosses bid which triggers a transaction. The sell MO eats up the buy LO according to price priority followed by time priority (of LO at the same price).
Events in the order book

- New limit sell order
- New limit buy order
- Market sell order
- Market buy order
- Cancellation of existing limit orders

Each market participant can choose between these actions on virtually all security exchanges. Sometimes additional events exist: Hidden limit orders etc.
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The market reacts to all of these events! **Impact** of limit order book events.
Market microstructure

Theory and consequences of the impact of order book events: Determinants of the bid-ask spread, statistical arbitrage, etc. I will give a short illustration of some of these aspects. Visible part of the limit order book.

Market “mesoscopic” microstructure

Behaviour of the market when large volumes are bought or sold. “Coarse-grained” approaches. Study of underlying true supply and demand curves: Hidden (and much bigger) “part” of the limit order book.
Market microstructure

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Market “mesoscopic” microstructure

Behaviour of the market when large volumes are bought or sold. “Coarse-grained” approaches. Study of underlying true supply and demand curves: Hidden (and much bigger) “part” of the limit order book.

Provocative statement

The publicly visible limit order book does not reflect the true supply and demand of a financial market!
Some important aspects of microstructure
The liquidity paradox

Liquidity providers submit limit orders. They want to trade.

But, liquidity providers do not want to disclose their private information!

Hence, liquidity providers do not display their buying/selling intentions.
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Hence, liquidity providers do not display their buying/selling intentions.

Still not convinced?

The total displayed volume in an order book of a liquid stock is about $\sim 0.1\%$ of the daily total traded volume! If you want to buy, for instance, 1.2% of a large company, there is no way you can achieve this at once!
The meta-order

So how do people trade large quantities?
The meta-order

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A meta-order is the *ensemble* of trades, executed *incrementally*, that belong to one single trading decision \{buy/sell, \(Q\}\).
Consequence 1: correlated order sign flow

Most trades belong to some meta-order. Hence, their sign is correlated with the signs of past and future trades!

Define \( \epsilon(t) = \pm 1 \) for buy/sell market order at time \( t \). Then \textit{empirically}

\[
\mathbb{E}[\epsilon(t)\epsilon(t+\tau)] \sim \tau^{-0.5}.
\]

This is a \textit{long memory} process and not in conflict with price efficiency (observed price is an almost perfect martingale).

Buys tend to be followed by buys and sells tend to be followed by sells.
Consequence 2: The bid-ask spread

Submit a *random* single BUY market order of volume $q$. It executes sell limit orders up to volume $q$, thereby the ask is *mechanically* impacted (moves up). Hereafter the ask relaxes (on average!) and mean-reverts to the initial price. The mid-price is virtually not impacted.
Consequence 2 : The bid-ask spread

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- hereafter the ask relaxes (on average!) and mean-reverts to the initial price. The mid-price is virtually not impacted.

But : Observed trades are immersed in a correlated order flow! Observed impact of trade takes into account impacts of all future correlated trades.
Consequence 2: The bid-ask spread

But: Observed trades are immersed in a correlated order flow! Observed impact of trade takes into account impacts of all future correlated trades.

Observe a single market buy order of volume $q$. It executes sell limit orders up to volume $q$,

- thereby the ask is *mechanically* impacted (moves up).
- hereafter the ask does not relax (on average!); the mid-price moves further up until reaching a plateau. One observes:

$$R = \sup_{t > 0} \mathbb{E}[p(t) - p(0) \mid \text{Buy MO at } t = 0] > 0.$$
Consequence 2: The bid-ask spread

\[ R = \sup_{t > 0} E [p(t) - p(0) \mid \text{Buy MO at } t = 0] > 0. \]

Apply competitive equilibrium methods for the liquidity provider. Conditioned on the fact that his/her limit order is executes, he/she makes average P&L of \( \mathcal{P} = \langle s \rangle / 2 - \langle R \rangle : \)

\[ \langle s \rangle / 2 - \langle R \rangle \to 0^+. \]

The relation \( \langle s \rangle \approx 2\langle R \rangle \) is well satisfied on liquid financial markets!
Mesoscopic microstructure
Back to our meta-order

Such a meta-order is defined by its *execution/trading rate*:

\[ m(t) = \frac{\partial}{\partial t} \text{inventory}(t). \]

Its *volume*:

\[ Q = \int_{t_-}^{t_+} dt \ m(t), \]

the *starting and ending times*

\[ t_- = \inf \text{supp}(m) , \quad t_+ = \sup \text{supp}(m). \]

\( T = t_+ - t_- \) is its *length*. 
The impact costs

\[ C = \int_{t_-}^{t_+} dt \ m(t) \mathbb{E}[p(t)] . \]
The impact costs

\[ C = \int_{t_-}^{t_+} dt \ E[p(t)] \cdot m(t) \).

We need to know how the price \( p(t) \) reacts to the meta-order!
The impact costs

\[ C = \int_{t_-}^{t_+} \, dt \, m(t) \mathbb{E}[p(t)]. \]

We need to know how the price \( p(t) \) reacts to the meta-order!

Introduce:

- peak impact \( \mathbb{E}[p(t_+) - p(t_-)] \) of meta-order.
- impact trajectory \( \mathcal{I}(t) = \mathbb{E}[p(t + t_-) - p(t_-)]. \)
Empirical formula for the peak impact

- peak impact $I(T) = E[p(t_+) - p(t_-)]$ of meta-order.
- impact trajectory $I(t) = \mathbb{E}[p(t + t_-) - p(t_-)]$.

Empirically, we find for the peak impact:

$$I(T) \approx Y \sigma \sqrt{\frac{Q}{ADV}},$$

with:

- $Y \sim 1$,
- $\sigma$ the daily volatility of the security,
- $ADV$ the average daily traded volume of the security.
The square-root law

\[ I(T) \approx Y \sigma \sqrt{\frac{Q}{ADV}}, \]

- Impact is non-linear in the volume $Q$.
- Impact is non-Markovian: Remember that the volume $Q$ is executed incrementally. Each single trade cannot have the same impact, otherwise total impact would be linear in $Q$. The first trade has larger impact than subsequent trades!
- Impact is approximately independent of the execution path and depends only on the total volume $Q$. A more detailed study does show variations with respect to the execution path: Optimal execution problem (later).
Towards a theory of market impact: Remember the liquidity paradox

Remember: Liquidity providers hide their intentions, they offer far less liquidity as is needed to execute large volumes. The consequences of this behaviour can be directly observed on financial markets: Existence of meta-orders, correlated order flow, etc.
Concept of the latent order book

Remember: Liquidity providers hide their intentions, they offer far less liquidity as is needed to execute large volumes. The consequences of this behaviour can be directly observed on financial markets: Existence of meta-orders, correlated order flow, etc.

True supply and demand is *not* displayed in the order book. There is a fictitious non-public book which records the intentions of market participants: The *latent order book*.
Concept of the latent order book

True supply and demand is *not* displayed in the order book. There is a fictitious non-public book which records the intentions of market participants: The *latent order book*. 

\[
\rho_a \rho_b \\
\text{p(t)}
\]
Dynamics of the latent order book

At each time step a fraction $\rho$ of all latent orders at each point $y$ moves with $dW_t$,

$$dW_t dW_s = \sigma^2 \delta(t - s) dt .$$

**Explanation:** This models exogeneous information instantly digested by a part $\rho$ of the market participants.

Define the ask and bid densities of the latent order book : $\rho_a(y, t)$ and $\rho_b(y, t)$. Set $\varphi(y, t) = \rho_a(y, t) - \rho_b(y, t)$.

Define $\phi(x, t) = \varphi(y, t)$ with $x = y - \rho \int^t dW_s$.

**Explanation:** This is a change of reference frame. We model the midprice $p(t) = \rho \int^t dW_s$ as a Brownian motion, $\phi$ is then the latent order book observed in the co-moving reference frame of $p(t)$, i.e. $x$ measures the relative distance with respect to $p(t)$. 

Dynamics of the latent order book

Define $\phi(x, t) = \varphi(y, t)$ with $x = y - \rho \int^t dW_s$.

\[
\dot{\varphi}(y, t) = -\rho dW_t \varphi'(y, t) + \frac{\rho}{2} \sigma^2 \varphi''(y, t),
\]

\[
\dot{\phi}(y, t) = \dot{\phi}(x, t) - \rho dW_t \phi'(x, t) + \frac{\rho^2}{2} \sigma^2 \phi''(x, t).
\]

This yields

\[
\dot{\phi}(x, t) = \frac{\sigma^2}{2} \rho (1 - \rho) \phi''(x, t).
\]

Set

\[
D = \frac{\sigma^2}{2} \rho (1 - \rho).
\]

$\phi$ satisfies a simple diffusion equation!
Market clearing in the latent order book

Note that: Market clearing can only occur in the real book!

Define: \( R(x) = r \rho_a(x)1_{x<0} + r \rho_b(x)1_{x>0} \), with \( r \) the reaction rate.

Explanation: The reaction rate corresponds to the fraction of latent orders, on the wrong side of the mid-price, that is submitted as (real) limit or market orders at the mid-price per unit time. The number of executed orders per unit time must be equal to \( J/D \) in the stationary case.
Market clearing in the latent order book

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Explanation: The reaction rate corresponds to the fraction of latent orders, on the wrong side of the mid-price, that is submitted as (real) limit or market orders at the mid-price per unit time. The number of executed orders per unit time must be equal to $J/D$ in the stationary case.

We have already defined $\phi(x) = \rho_a(x) - \rho_b(x)$ (with $x$ measured in the co-moving reference frame). We need to study the second superposition $\psi(x) = \rho_a(x) + \rho_b(x)$.

Objective: The solution $\psi(x)$ depends on the inhomogeneity $R(x)$. Since $R(|x|) = r[\psi(x) - \phi(|x|)]$ we find a self-consistent equation for $R(x)$. 
Market clearing in the latent order book

Solution:

\[
R(x) \sim |x| K_1(\sqrt{J/2D} \cdot |x|) \approx \begin{cases} 
\text{const.} & |x| \to 0 \\
 e^{-\sqrt{J/2D} \cdot |x|} & |x| \to \infty 
\end{cases}
\]

The overlap has a finite width \( \sqrt{D/J} \). The amplitude of the overlap scales with \( 1/r \).

Hence: When \( r \to \infty \) the overlap vanishes independently of the available liquidity \( \mathcal{L} = J/D \).
Market clearing in the latent order book

The overlap has a finite width $D/J$. The amplitude of the overlap scales with $1/r$. Hence: When $r \to \infty$ the overlap vanishes independently of the available liquidity $L = J/D$.

The market clearing mechanism thus works according to:

- The price movements lead to latent outstanding liquidity (latent buy orders above and latent sell orders below the current market price).
- These orders cannot be instantly executed due to restricted liquidity. Instead, traders disclose at each time step a fraction $r \Delta t$ of the outstanding orders and submit it at real market/limit orders.
- This mechanism leads to market clearing in the latent book: When $r$ is big, outstanding liquidity becomes small such that the incoming flow of new orders is kept constant. Hence, the limit $r \to \infty$ is commensurate with restricted liquidity in the real book.
- The current market price and the latent market clearing price coincide!
Towards the price impact formula

Homogeneous problem with boundary conditions:

\[ \partial_t \phi(x, t) = D \partial_{xx} \phi(x, t), \]
\[ \lim_{x \to \pm \infty} \partial_x \phi(x, t) = J. \]
Towards the price impact formula

Consider an additional trader who wishes to execute his metaorder. Non-homogeneous problem:

\[ \partial_t \phi(x, t) - D \partial_{xx} \phi(x, t) = m(t) \delta(x - x_t), \]

\[ \lim_{x \to \pm \infty} D \partial_x \phi(x, t) = J, \]

where \( m(t) \) is the trading rate and \( x_t \) is the mid-price, defined via

\[ \phi(x_t, t) = 0. \]

\( (\text{market clearing}) \)
The price impact formula

Solution:

$$\phi(x, t) = -\frac{J}{D}x + \int_{-\infty}^{\infty} dy \int_{0}^{t} dt' \frac{m(t')}{\sqrt{4D\pi(t - t')}} e^{-\frac{(x - y)^2}{4D(t - t')}} \delta(y - x_{t'}).$$

Integrate over the Dirac-distribution and use that $x_t$ is a zero of $\phi$:

$$x_t = \frac{D}{J} \int_{0}^{t} dt' \frac{m(t')}{\sqrt{4D\pi(t - t')}} e^{-\frac{(x_t - x_{t'})^2}{4D(t - t')}}.$$

Implicit equation for $x_t$ which depends on the whole history $\{x_{t'} < t\}$. 
**Example of impact**

Constant $m$ and relaxation after meta-order is completed. With additional noise in the order book.
Example of impact

Constant $m$ and relaxation after meta-order is completed. With additional noise in the order book.
Absence of dynamical arbitrage

Define the execution cost:

\[ C[m] = \int_0^T \int d t \ m(t)x_t. \]

Then we have the following theorem:

Let \( x_t \) be the solution of our model for some non trivial round-trip \( m \), i.e. \( \int dt \ m(t) = 0 \) and \( m \neq 0 \). Then, execution costs are strictly positive, \( C > 0 \).
Let \( x_t \) be the solution of our model for some \textit{non trivial round-trip} \( m \), i.e. \( \int dt \ m(t) = 0 \) and \( m \neq 0 \). Then, execution costs are strictly positive, \( C > 0 \).

\textbf{One-line proof :}

Use the price impact formula of our model to replace \( x_t \):

\[
C = \frac{1}{2} \int_0^T \int ds \ M(t, s) m(s),
\]

with

\[
M(t, s) = \frac{D^2}{J} \int_{-\infty}^{\infty} dz \ du \ K(t, u; z) K^*(s, u; z),
\]

\[
K(t, u; z) = z \cdot 1_{u \leq t} \cdot e^{-Dz^2(t-u)+izx_t}.
\]

Hence \( M \) is a square and thus positive.
Square-root solution for constant trading rate

\[ x_t = A\sqrt{Dt} \]

is an exact solution with

\[ A = \frac{m}{J} \int_0^1 d\eta \frac{e^{-\frac{A^2(1-\sqrt{\eta})}{4(1+\sqrt{\eta})}}}{\sqrt{4\pi(1-\eta)}}. \]
Square-root solution for constant trading rate

\[ A = \frac{m}{J} \int_0^1 \, d\eta \, \frac{e^{-\frac{A^2(1 - \sqrt{\eta})}{4(1 + \sqrt{\eta})}}}{\sqrt{4\pi(1 - \eta)}}. \]

is such that for large \( m \) impact is square-root in volume since \( A^2 \approx 2m/J \):

\[ x_T \sim \sqrt{Q}. \]
Square-root solution for constant trading rate

\[ A = \frac{m}{J} \int_0^1 d\eta \frac{e^{-\frac{A^2(1-\sqrt{\eta})}{4(1+\sqrt{\eta})}}}{\sqrt{4\pi(1-\eta)}}. \]

is such that for large \( m \) impact is square-root in volume since \( A^2 \simeq 2m/J \):

\[ x_T \sim \sqrt{Q}. \]

The model reproduces the empirically observed square-root law. Impact is square-root not only at its peak but throughout the whole impact trajectory.
Beyond the square-root law: Small constant trading rate

\[ A = \frac{m}{J} \int_0^1 d\eta \frac{e^{-\frac{A^2(1-\sqrt{\eta})}{4(1+\sqrt{\eta})}}}{\sqrt{4\pi(1-\eta)}}. \]

is such that for small \( m \) impact is with \( A \approx \frac{m}{J} : \)

\[ x_T \sim \sqrt{mQ}. \]

The model predicts vanishing impact for small trading rates \( m \)! Not accounted for in the empirical formula.
(Non-)Mechanical impact

Our model predicts the mechanical impact, i.e. the impact of a meta-order *without* any information content (random meta-order).

If the meta-order bears information, an additional non-mechanical “impact” arises (not really an impact as not stimulated by the meta-order itself but by its conditioning to the direction of the market).

After completion of the meta-order, mechanical impact decreases to zero. However, non-mechanical impact remains positive if order bears information.
Impact of “informed” and “random” meta-order

\[ \mathbb{E}[p(t+t) - p(t)] \]

permanent impact
Problem of finding the “bare” mechanical impact

How to “subtract” the information from the empirically observed impact to obtain the mechanical impact?

Answer:

Subtract permanent impact from observed peak impact.
Empirical analysis on the Bitcoin/USD exchange market
Empirical analysis : What has been done?

- Square-root law holds on all stock/future markets (the “empirical formula”). Permanent impact is \( \approx \frac{2}{3} \) of peak impact (2/3-law).
- No convincing analysis of market impact for small \( m \). The “subtraction trick” has not been used in the past.
- Empirical analysis is difficult : Insider knowledge is necessary to reconstitute the meta-orders from anonymous trades.
Empirical analysis

We have a complete dataset of 2 million meta-orders on the BitCoin/USD exchange market

- Square-root law holds on the BitCoin, despite absence of sophisticated arbitrage ( !)
- Mechanical impact vanishes for small trading rate $m$! (no good picture, yet)
Square-root law on the Bitcoin: even trajectory-wise!

Trading rate: 10 BTC/s

Trading rate: 3 BTC/s
Impact of informed/uninformed meta-orders on the Bitcoin
Summary of empirical evidence
Summary of empirical evidence: BitCoin, futures and stocks

- Impact is square-root: Peak and trajectory. Hence, impact is non-linear and non-Markovian.
- Subtraction trick shows that small trading rates lead to smaller impact at variance with the standard empirical formula and in agreement with theoretical model based on latent liquidity.
- Permanent impact is $\approx \frac{2}{3}$ of peak impact.
- Uninformed trades (uncorrelated with the underlying orderflow; cashflow trades) do not have any permanent impact. Hence, permanent impact is a phenomenon solely based on the information content of the meta-order.
Applications and further aspects
Application 1: Optimal execution

Liquidation horizon $T$, quantity $Q$ fixed. $\mathcal{F}$ space of possible execution rates $m : [0, T] \rightarrow \mathbb{R}$.

$$ C : \mathcal{F} \rightarrow \mathbb{R}^+ $$

$$ \arg \max_{m \in \mathcal{F}} C[m] ? $$

All examples in the literature treat optimal execution within a Markovian setup:

$$ \mathbb{E}[p_{t+dt} | p_t] = \text{const.} \cdot dt \cdot f(m_t) . $$

Even if $f(m)$ is non-linear, above impact model does not correspond to reality!

Generalization to non-Markovian impact: Ongoing work!
Application 2: Stock pinning

*Empirical observation:* Spot price tends to converge to one of the possible strike prices on stocks with heavy derivative use such as Apple and Google.

*Assumption 1:* Option sellers (banks) $\Delta$-hedge, buyers (industry companies) do not (or less).

*Assumption 2:* Impact is *linear*: $\dot{x}_t = L^{-1} \dot{Q}$ (*major contradiction with our work*).

Banks hold $-C_t + \Delta(t, x_t)x_t$ and

$$\mathcal{L}\dot{x}_t = \frac{d}{dt} \Delta(t, x_t) = \dot{\Delta} + \dot{x}_t \Delta'$$

$$\Rightarrow \dot{x}_t = \frac{\partial \Delta(t, x_t)/\partial t}{\mathcal{L} - \partial \Delta(t, x_t)/\partial x}.$$ 

$\dot{\Delta} \geq 0$ for $x_t \geq S$ and $\lim_{t \to T} \Delta'(t, S) \to \infty$. Hence, small enough $T - t$ there the strike is a local attractor!

*How to generalize to non-linear impact?*
Appendix 1: Farmer-Lillo-Gerig-Waelbroeck-model and the 2/3-law

Objective: Calculate impact profile during the execution of a meta-order of length $T$

$x_t$ is the execution price. Then

$$x_{t+1} = \begin{cases} 
  x_t + s_t^+ \\
  x_t - s_t 
\end{cases}$$

$P_t = \text{Probability at } t \text{ that meta-order continues at } t$

$$= \mathbb{P}[T > t | T > t-1] = \frac{\mathbb{P}[T \geq t+1, T \geq t]}{\mathbb{P}[T \geq t]} = \frac{\mathbb{P}[T \geq t+1]}{\mathbb{P}[T \geq t]}.$$
Farmer-Lillo-Gerig-Waelbroeck-model and the $2/3$-law

The martingale condition

$$s_t^- (1 - P_t) = s_t^+ P_t.$$  

$$x_t = \mathbb{E}[x_\infty \mid t] = (1 - P_t) \mathbb{E}[x_\infty \mid \text{MO stops at } t] + P_t(x_t + s_t^+).$$

The fair-pricing condition

$$\mathbb{E}[x_\infty \mid \text{MO stops at } t] = \frac{1}{t} \sum_{k=0}^{t} x_k.$$  

(All trades have unit volume).
Farmer-Lillo-Gerig-Waelbroeck-model and the $2/3$-law

We find:

$$tx_t = ts_t^- + \sum_{k=0}^{t} x_k,$$

and finally

$$\frac{t-1}{t} \frac{1-P_t}{1-P_{t-1}} s_{t-1}^+ = P_t s_t^+.$$

Note that empirically $\mathbb{P}[T \geq t] \sim t^{-\gamma}$ with $\gamma \approx 3/2$. For large $t \gg 1$ we have

$$s_t^+ \sim \left(1 - \frac{2-\gamma}{t}\right) s_{t-1}^+ \sim t^{\gamma-2} s_0^+$$

$$x_t \sim t^{\gamma-1} \sim \sqrt{t}$$

$$\mathbb{E}[x_\infty | \text{MO stops at } t] \sim \frac{2}{3} x_t.$$
Thank you!