Optimal Investment and Consumption with Small Transaction Costs

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Introduction

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Introduction

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Introduction

Portfolio Choice with Frictions

- Optimal portfolio selection is a key problem in finance.
  - Individual decision making.
  - Starting point for equilibrium models.
  - Not feasible with frictions.
- Optimal behavior should reflect tradeoff between:
  - Displacement from optimal allocation.
  - Costs of trading.
- Important?
- When and how?
- Simple and robust adjustments?
Introduction
Passive Investment

- This talk: *proportional* costs. Bid-ask spreads.
- Key insight (Magill/Constantinides, 1976):
  - *No-trade region* around the frictionless target.
  - Remain inactive while inside.
  - Start trading when boundaries are breached.
- Mathematically precise formulation?
  - Singular control. Reflected diffusions à la Harrison.
- Numerical results (Constantinides, 1986):
  - Even small costs have large effect on asset demand.
  - But welfare loss is small if trading is reduced optimally.
- Assumes constant investment opportunities.
  - Strategies almost passive. Rebalancing is only motive to trade.
  - What about more active trading strategies?
Introduction

Active Investment

- Complex models typically intractable with frictions.
- Numerical results in concrete settings:
  - Frictions have much bigger effect.
  - But difficult to understand structure and comparative statics.
- Alternative: asymptotics for small costs.
  - Treat problem as perturbation of its frictionless counterpart.
  - Compute leading-order corrections of optimal policy and performance.
- Recent progress for active investment strategies.
Introduction
Active Investment ct’d

- Hedge a call option in Black-Scholes framework.
- Maximize sum of one-period mean variance profits.
- Infinite horizon investment/consumption problems.
- Different concrete models and objectives.
- This talk: pass to general setting.
  - Uncovers underlying general structure.
  - Resulting formulas are easy to interpret and implement. Robust with respect to particular model specifications.
Results
Model

General asset prices:

- One safe asset. Normalized to one.
- One risky asset. Traded with proportional costs $\varepsilon_t = \varepsilon E_t > 0$.

Mid price:

$$dS_t = b_t^S dt + \sqrt{c_t^S} dW_t$$

- General diffusive dynamics.
- Can include heteroskedasticity and predictable returns leading to market timing.
- No Markovian structure required.
- Transaction costs can be random and time-varying. 
  \text{Itô process } E_t \text{ rescaled by small parameter } \varepsilon.
Results

Model

General investment/consumption problem:

- Investor solves:

\[
E \left[ \int_0^T u_1(t, \kappa_t^\varepsilon) dt + u_2(X_T^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon)) \right] \rightarrow \text{max!}
\]

over policies \((\varphi_t^\varepsilon, \kappa_t^\varepsilon)\) with wealth processes

\[
X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) = X_0 + \int_0^t \varphi_s^\varepsilon dS_s - \int_0^t \kappa_s^\varepsilon ds + \Psi_t - \int_0^t \varepsilon_s d||\varphi^\varepsilon||_s
\]

- Utility from intermediate consumption and terminal wealth.
- Random endowment stream \(\Psi_t\).
- Covers hedging, lifecycle investing, market timing, etc.
Adjustment of the frictionless optimal policy (Kallsen/M-K, 2014):

- Use frictionless consumption.
  - Robust. Only adjust for reduced wealth.
- Time- and state-dependent no-trade region:
  \[ [\overline{NT}_t - \Delta NT_t, \overline{NT}_t - \Delta NT_t] \]
- Midpoint \( \overline{NT}_t \) is frictionless target.
  - Also only adjusted for reduced wealth.
- Half-width \( \Delta NT_t \) is the crucial quantity:
  \[
  \Delta NT_t = \left( \frac{3R_t}{2} \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t \right)^{1/3}
  \]
Results
Approximately Optimal Policy ct’d’d

- Half-width of optimal no-trade region:

\[
\left( \frac{3R_t}{2} \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t \right)^{1/3}
\]

- Key driver: *portfolio gamma* \( d\langle \varphi \rangle_t / d\langle S \rangle_t \).
  - Ratio of squared diffusion coefficients.
  - Active strategies require wide buffer.
  - Turbulent markets call for close tracking.
  - For delta-hedge: option gamma.
  - Sample from realized variance of frictionless benchmark.

- Only current spread \( \varepsilon_t \) matters for correction.
  - Future dynamics not hedged at the leading order.

- Preferences subsumed by *indirect risk-tolerance* \( R_t \).
Measure for risk tolerance?

- Risk tolerance \( R_t = -\frac{U'(t,X_t)}{U''(t,X_t)} \) of the indirect utility:

\[
U(t,x) = \sup_{(\varphi,\kappa)} E_t \left[ \int_t^T u_1(s,\kappa_s)ds + u_2 \left( x + \int_t^T \varphi_s dS_s - \int_t^T \kappa_s ds \right) \right]
\]

- Current against future consumption. Average over scenarios.
- Quantifies wealth-dependence of preferences.
- Bound to appear in any perturbation of frictionless problems.
  - Utility-based prices and hedges for small claims (Kramkov/Sîrbu, 2006).
  - Sensitivity of optimal consumption streams w.r.t. perturbations of the endowment (Herdegen/M-K, 2015).
- Here: novel dynamic characterization by quadratic BSDE.
Results

Performance

- Performance loss due to trading costs?
- Maximal utilities: \( U(x) \) without and \( U^\varepsilon(x) \) with costs.
- Certainty equivalent loss (Kallsen/M-K, 2014):

\[
U^\varepsilon(x) \sim U \left( x - E^Q \left[ \int_0^T \frac{(\Delta N T_t)^2}{2R_t} d\langle S \rangle_t \right] \right)
\]

- Portfolio gamma \( d\langle \varphi \rangle_t / d\langle S \rangle_t \) quantifies liquidity risk. Appealingly robust proxy: also central for..
  - ..discrete trading (Bertsimas, Kogan, Lo, 2000; Hayashi/Mykland, 2005)
  - ..optimal discretization (Fukasawa, 2011, 2013; Rosenbaum/Tankov, 2014)
  - ..other trading costs (Altarovici, M-K, Soner, 2013; Moreau, M-K, Soner, 2014)
Performance loss:

- Portfolio gamma $d\langle \varphi \rangle_t/d\langle S \rangle_t$ determines magnitude.
  - Transaction costs matter for *active* trading!
- *Universal* scaling for welfare effect of small costs:
  - Two thirds caused by trading costs.
  - One third by displacement.
- For small transaction *tax* in the spirit of Tobin:
  - Two thirds of welfare loss paid to government. Can be redistributed.
  - One third dissipates. True “friction”.
- Result surprisingly robust. Independent of asset price and cost dynamics, preferences, endowments.
- Only assumptions: diffusive prices, proportional cost structure.
Results
General Equilibrium

So far: partial equilibrium models. General equilibrium?

- Needed to analyze policies like a financial transaction tax.
  - 2/3-1/3 split of welfare losses robust?
- Finance literature: numerical solution of discrete models.
  - Asymptotic analysis of a particular model with fixed costs.
  - Bank account exogenous. No full equilibrium.
- Current work in progress with Martin Herdegen:
  - Endogenous asset returns and interest rates.
  - Linear transaction tax paid to state. Consumes optimally.
Results

General Equilibrium ct’d

Effect of a *small* friction?

- Assume all agents use leading-order optimal strategies.
- No-trade regions have to match for stock market clearing:
  - Midpoints offset for exponential utilities.
  - Also need
    \[
    \left( \frac{3R^1_t}{2} \frac{d\langle \varphi^1 \rangle_t}{d\langle S \rangle_t} \varepsilon^1_t \right)^{1/3} = \left( \frac{3R^2_t}{2} \frac{d\langle \varphi^2 \rangle_t}{d\langle S \rangle_t} \varepsilon^2_t \right)^{1/3}
    \]
  - Frictionless market clearing implies \( d\langle \varphi^1 \rangle_t = d\langle \varphi^2 \rangle_t \).
  - Split of tax \( \varepsilon = \varepsilon^1_t + \varepsilon^2_t \) determined by risk tolerances \( R^1_t, R^2_t \).
- Consumptions of agents and state need to clear bond market.
  - Can be ensured using sensitivity analysis of optimal consumption streams (Herdegen/M-K, 2015).
- Final result: frictionless equilibrium robust.
  - Does not need to change because of small friction.
  - Partial equilibrium analysis justified for exponential utilities.
Summary

- Approximately optimal policy with small proportional transaction costs.
  - “Myopic” correction for small frictions.
  - Drivers: current trading cost, indirect risk tolerance, portfolio gamma.
- Leading order welfare loss.
  - 2/3 due to trading costs, 1/3 due to displacement.
  - Portfolio gamma $\frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t}$ quantifies liquidity risk.
- Results are very robust.
  - General preferences, price and cost dynamics.
  - Random endowments.
  - No Markovian structure required.
  - Results extend to other optimization criteria and frictions.
  - Extension to general equilibrium for exponential utilities.
Derivations
Small-Cost Expansion

- How to derive the results summarized above?
  - General, non-Markovian, singular control problem.
  - Where do the myopic small-cost corrections come from?

- Let us sketch the idea on an informal level.
- For simplicity:
  - Utility from terminal wealth only.
  - Constant absolute risk tolerance $R = -u'_2 / u''_2$.

- Perform second-order Taylor expansion around the frictionless optimal wealth process $x + \int_0^T \varphi_t dS_t$.

- Two perturbations:
  - Small trading cost $\varepsilon_t$.
  - Small adjustment $\Delta \varphi_t$ of the trading strategy.
Derivations
Transaction Costs and Displacement

\[
E \left[ u_2 \left( x + \int_0^T (\varphi_t + \Delta \varphi_t) dS_t - \int_0^T \varepsilon_t d||\varphi + \Delta \varphi||_t \right) \right]
\approx E \left[ u_2 \left( x + \int_0^T \varphi_t dS_t \right) \right]
+ \beta E_Q \left[ \int_0^T \Delta \varphi_t dS_t - \int_0^T \varepsilon_t d||\varphi + \Delta \varphi||_t \right]
- \frac{1}{2} \beta E_Q \left[ R^{-1} \left( \int_0^T \Delta \varphi_t dS_t - \int_0^T \varepsilon_t d||\varphi + \Delta \varphi||_t \right) \right]^2
\]

- Here: Q is the frictionless dual martingale measure with density \( dQ/dP = u'_2(x + \int_0^T \varphi_t dS_t) / \beta \).
Derivations

Transaction Costs and Displacement ct’d

Whence:

\[
E \left[ u_2 \left( x + \int_0^T (\varphi_t + \Delta \varphi_t) dS_t - \int_0^T \varepsilon_t d||\varphi + \Delta \varphi||_t \right) \right]
\approx E \left[ u_2 \left( x + \int_0^T \varphi_t dS_t \right) \right] - \beta E_Q \left[ \int_0^T \varepsilon_t d||\varphi + \Delta \varphi||_t \right]
- \frac{1}{2} \beta E_Q \left[ R^{-1} \int_0^T (\Delta \varphi_t)^2 d\langle S \rangle_t \right]
\]

First correction term represents expected transaction cost loss.
Second corresponds to displacement loss.
Computation?
Derivations
Homogenization

- Ansatz: optimal strategy remains close to frictionless target by reflection off trading boundaries.
- Whence: deviation follows reflected diffusion.
- Change of time, space: approximate by reflected Brownian motion with infinitesimal variance $d\langle \varphi \rangle_t/dt$ at the first order.
- Transaction costs $= $ local time at boundaries.
  - Expectation given by $(d\langle \varphi \rangle_t/dt)/2\Delta N T_t$.
- Stationary law uniform.
  - Ergodic theorem allows to replace squared deviation $\Delta \varphi_t^2$ by expectation $\Delta N T^2/3$.
- Separation of time scales. Fast variable is “homogenized” out.
Derivations

Pointwise Optimization

In summary:

- **Transaction cost loss:**

  \[ \beta E_Q \left[ \int_0^T \epsilon_t \frac{d <\varphi>_t}{dt} \frac{dt}{2\Delta N T_t} \right] \]

- **Displacement loss:**

  \[ \frac{\beta}{3R} E_Q \left[ \int_0^T \Delta N T_t^2 \frac{d <S>_t}{dt} \frac{dt}{dt} \right] \]

- **Optimal boundaries determined by pointwise maximization:**

  \[ \Delta N T_t = \left( \frac{3R}{2} \frac{d <\varphi>_t}{d <S>_t} \epsilon_t \right)^{1/3} \]
Other trading costs? (⇝ tonight’s talk)
  ▶ Basic idea similar. But renormalized deviations differ:
    ▶ Reflected Brownian motion for proportional costs.
    ▶ Fixed costs: killed Brownian motion restarted at the origin.
    ▶ OU-type process with quadratic costs.
  ▶ Trading costs scale differently.
  ▶ Asymptotic stationary law depends on control used:
    ▶ Uniform for proportional costs.
    ▶ “Hat function” for fixed costs.
    ▶ Gaussian for quadratic costs.

For papers and preprints:
  ▶ http://www.math.ethz.ch/~jmuhleka/