Continuous Time Perpetuities and the Time Reversal of Diffusions

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Research Goal

Identify the *distribution* of the perpetuity

\[ X_0 := \int_0^\infty D_t f(Z_t) dt. \]

- \( D \): discount process (potentially stochastic).
- \( f(Z) \): cash flow rate.
- \( Z \): stationary ergodic diffusion ("economic factors").

In fact, we seek

- The joint distribution of \((Z_0, X_0)\).
- The conditional distribution of \(X_0\) given \(Z_0 = z\).
Outline

Problem Setup

Literature Review

Main Results

- Construction of an ergodic process with limiting distribution equal to the joint distribution of $(Z_0, X_0)$.

- PDE for the conditional cumulative distribution function of $X_0$ given $Z_0 = z$. 
Problem Setup: \( X_0 = \int_0^\infty D_t f(Z_t) dt \)

We assume:

- \( Z \): stationary ergodic diffusion taking values in \( E \subseteq \mathbb{R}^d \).
  \[
dZ_t = m(Z_t)dt + \sigma(Z_t)dW_t; \quad Z_0 \sim p \perp W.
\]

- Local ellipticity: \( c = \sigma \sigma' > 0 \) in \( E \).
- \( p \): invariant density.
- \( D \): stochastic discount factor.
  \[
  D_t = e^{-\int_0^t a(Z_u)du} \mathcal{E} \left( -\int_0^t \theta' \sigma(Z_u) dW_u - \int_0^t \eta(Z_u)' dB_u \right)
  \]

- \( B \): Brownian motion. \( B \perp W, Z_0 \).
- \( D \) decays: \( e^{\kappa t} D_t \to 0 \) almost surely for some \( \kappa > 0 \) and all starting points \( z \in E \) for \( Z_0 \).

- \( f \): Borel measurable, \( f \geq 0 \). \( f \in L^1(p) \) (e.g. \( f \) bounded).
Obtaining the distribution of $X$ is important in Actuarial Science, Mathematical Finance.

- [Duf90]: $X_0 = \int_0^\infty e^{-\nu t - B_t} dt$. $X \sim \text{inv. Gamma}$.

- [Del93, Yor01]: $X_0 = \int_0^\infty e^{-\int_0^t Z_u du} dt$, $Z \sim \text{CIR}$.
  - Obtain moments, moment bounds.

- [Pau97, PH99]: $X_0 = \int_0^\infty e^{-\int_0^t Z_u du} dt$. $Z$ diffusion.
  - PDE for Laplace transform. Recurrent chain with invariant distribution of $X$ as limiting distribution.

- [Pau93, NP96, GP97, CPY01]: $X_0 = \int_0^\infty e^{-R_t} dP_t$, $R \perp \perp P$ Lévy processes.
  - Explicit identification of distribution through Laplace transform/Characteristic function.
[Duf92]: time reversal in discrete time.

- Provide argument to give insight.

\[ X_0 = \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} D_j \right) f_n. \]

- \((D_n) \perp \perp (f_n)\), each i.i.d.

For each \( N \) set \( X_N = \sum_{n=1}^{N} \left( \prod_{j=1}^{n} D_n \right) f_n. \)

Note that \( X_N \overset{\mathcal{L}}{=} \tilde{X}_N \) where

- \( \tilde{X}_N = D_N f_N + D_N D_{N-1} f_{N-1} + \ldots + (\prod_{j=1}^{N} D_j) f_1 \)

- Follows by re-ordering the independent components.
Literature Review

Recursive equations

\[ X_{N+1} = X_N + \left( \prod_{j=1}^{N+1} D_j \right) f_{N+1}. \]

\[ \tilde{X}_{N+1} = D_{N+1} \left( \tilde{X}_N + f_{N+1} \right) \]

\[ \text{Note: } D_{N+1} \perp \perp f_{N+1} \perp \perp \tilde{X}_N.\]

Reversed process much more manageable.

Taking \( N \uparrow \infty \), in law \( X \) satisfies

\[ X = D(X + f), \quad D \sim D_1, f \sim f_1, D \perp \perp f \perp \perp X. \]

\[ \text{Well studied object - see [Ver79] amongst many others.} \]
Main Results: $X_0 = \int_0^\infty D_t f(Z_t) dt$

$dZ_t = m(Z_t) dt + \sigma(Z_t) dW_t, \ Z_0 \sim p \perp W$

- $Z$ stationary, ergodic. $f \geq 0, f \in L^1(p)$.

$D_t = e^{-\int_0^t a(Z_u) du} \mathcal{E} (-\int_0^t \theta' \sigma(Z_u) dW_u - \int_0^t \eta(Z_u)' dB_u)_t$

- $B \perp W, Z_0. \ e^{\kappa t} D_t \to 0$ for some $\kappa > 0$ and all $Z_0 = z \in E$.

Set

- $\pi$: joint distribution of $(Z_0, X_0)$.
- $g$: conditional c.d.f. of $X_0$ given $Z_0 = z$.

Main Results:

- We use time-reversal to construct an ergodic process $(\zeta, \chi)$ with limiting distribution $\pi$.
- When $\eta' \eta > 0, f \in C^2(E)$, we prove $g$ solves a certain PDE on $E \times (0, \infty)$ (hence $X_0$ has a density on $(0, \infty)$).
Time Reversal Argument: Main Steps

Create a process out of $X$:
\[ X_t := \frac{1}{D_t} \int_t^\infty D_u f(Z_u) du \text{ for } t \geq 0. \]

Time-reverse $(Z, X)$ (fix $T > 0$):

\[ \zeta_T^T := Z_{T-t}, \chi_T^T := X_{T-t}. \]

Obtain dynamics of $(\zeta^T, \chi^T)$:

\[ \text{Create the “backwards” filtration } \mathcal{G}^T. \]
\[ \zeta^T: \text{ dynamics follow from } [HP86]: \]
\[ d\zeta_t^T = \tilde{m}(\zeta_t^T) dt + \sigma(\zeta_t^T) dW_t^T. \]
\[ \tilde{m}^i(z) = -m^i(z) + \frac{1}{p(z)} \sum_{j=1}^d \partial_j(c^j(z)p(z)) \text{ for } i = 1, \ldots, d. \]
\[ \tilde{m} = m \text{ if } Z \text{ is reversing (e.g. 1 dim.). Else, } \tilde{m} \text{ depends upon } p. \]
\[ \text{Manually obtain } \chi^T \text{ dynamics using } \mathcal{G}^T \text{ B.M. } B_t^T = B_T - B_t. \]
Time Reversal Main Steps

Dynamics for \((\zeta^T, \chi^T)\) (integrated form):

\[
\zeta^T_t = \zeta^T_0 + \int_0^t \tilde{m}(\zeta^T_u)du + \int_0^t \sigma(\zeta^T_u)dW^T_u
\]

\[
\chi^T_t = \Delta^T_t \left( \chi^T_0 + \int_0^T (\Delta^T_u)^{-1} f(\zeta^T_u)du \right)
\]

\[
\Delta^T_t := e^{-\int_0^t b(\zeta^T_u)du} \mathcal{E} \left( \int_0^t (\theta' c(\zeta^T_u)dW^T_u + \eta(\zeta^T_u)'dB^T_u) \right)_t
\]

(long formula for \(b(\zeta)\)) In differential form:

\[
d\zeta^T_t = \tilde{m}(\zeta^T_t)dt + \sigma(\zeta^T_t)dW^T_t.
\]

\[
d\chi^T_t = (f(\zeta^T_t) - \chi^T_t b(\zeta^T_t))dt + \chi^T_t (\theta' \sigma(\zeta^T_t)dW^T_t + \eta(\zeta^T_t)'dB^T_t).
\]

Generator \(L_R\) for \((\zeta^T, \chi^T)\) does not depend upon \(T\).

\[
\cdot \text{Consider a generic solution } (\zeta, \chi) \text{ (and } \Delta) \text{ with } L_R.
\]
Time Reversal Main Steps

Generic copies. Fix $x > 0$:

\[
\begin{align*}
\zeta_t &= \zeta_0 + \int_0^t \tilde{m}(\zeta_u) du + \int_0^t \sigma(\zeta_u) dW_u \\
\Delta_t &= e^{-\int_0^t b(\zeta_u) du} \mathcal{E}\left( \int_0^t (\theta' c(\zeta_u) dW_u + \eta(\zeta_u)' dB_u) \right) \\
\chi^x_t &= \Delta_t \left( x + \int_0^T (\Delta_u)^{-1} f(\zeta_u) du \right)
\end{align*}
\]

Dynamics

\[
\begin{align*}
\cdot d\zeta_t &= \tilde{m}(\zeta_t) dt + \sigma(\zeta_t) dW_t. \\
\cdot d\chi^x_t &= (f(\zeta_t) - \chi^x_t b(\zeta_t)) dt + \chi^x_t (\theta' \sigma(\zeta_t) dW_t + \eta(\zeta_t)' dB_t)
\end{align*}
\]

Drift for $\chi^x$ is mean-reverting for all $x$ (indep. of $x$).

\[
\cdot Z\ ergodic\ implies\ \zeta\ ergodic.\ With\ \chi^x\ mean\ revert,\ maybe\ (\zeta, \chi^x)\ is\ ergodic\ as\ well?
\]
Time Reversal Main Steps

Ergodicity, Case 1: \( \eta' \eta > 0 \) on \( E \) and \( f \in C^2(E) \).

- PDE argument shows \( \pi \) has a density on \( F = E \times (0, \infty) \) which satisfies \( L^*_R \pi = 0 \) where \( L^*_R \) is the formal adjoint to \( L_R \).
- \( \chi^x \) does not “explode” to \( 0, \infty \). Thus ergodicity with limiting density \( \pi \) holds ([Pin95]).
- Strong Law: for all \( (z, x) \in F \) and \( h \in B(F) \), if \( \zeta_0 = z \):
  \[
  \frac{1}{T} \int_0^T h(\zeta_t, \chi^x_t) dt \to \int_F h \, d\pi \text{ a.s.}
  \]

  - In fact, this can be strengthened to:
    - If \( \hat{\pi}^{z, x}_T \) is the empirical law for \( (\zeta, \chi^x) \) on \( [0, T] \) given \( \zeta_0 = z \), then for each \( z \in E \), \( \hat{\pi}^{z, x}_T \xrightarrow{\text{w}} \pi \) with prob. one for all \( x > 0 \).
Time Reversal Main Steps

Ergodicity Case 2: degenerate $\eta$, $f \in L^1(p)$.

- No PDE argument: $X_0$ may in fact contain an atom.
  - $X_0 = \int_0^\infty e^{-t} dt = 1$.

- Use a double perturbation argument:
  - Step 1. $f \in C^2(E)$. Approximate $\chi^x$ with
    \[
    \chi_{t,\epsilon,\chi} := \Delta_t E \left( \sqrt{\epsilon} \hat{B} \right)_t \left( x + \int_0^t \left( \Delta_u E \left( \sqrt{\epsilon} \hat{B} \right)_u \right)^{-1} f(\zeta_u)du \right).
    \]
    where $\hat{B}$ is an independent B.M. $\chi_{t,\epsilon,\chi}$ has $|\eta^{\epsilon}|^2 \geq \epsilon$. Invoke previous result then take $\epsilon \downarrow 0$.

- Step 2. Approximate $f \in L^1(p)$ with $f^n \in C^2(E)$:
  \[
  \int_E |f^n(z) - f(z)|p(z)dz \leq n^{-2}2^{-n}
  \]
  Invoke above result then take $n \uparrow \infty$. 
Time Reversal Main Steps

Ergodicity: Main Results (general $\eta, f$)

- Set $\hat{\pi}_T^x$ as the empirical law of $(\zeta, \chi^x)$ on $[0, T]$ given $\zeta_0 \sim p$.
- Then, with probability one

$$\hat{\pi}_T^x \xrightarrow{w} \pi \text{ for all } x > 0.$$  

- In other words, with probability one:

$$\frac{1}{T} \int_0^T h(\zeta_t, \chi^x_t) dt \to \int_F h d\pi \text{ for all } h \in C_b(F), x > 0.$$  

- Estimation of $\pi$ starting $\chi$ anywhere.
Estimation of $\pi$ from Simulation

In all cases, if $\zeta_0 \sim p$ with probability one

$$\frac{1}{T} \int_0^T h(\zeta_t, \chi^x_t) dt \to \int_F h d\pi \text{ for all } h \in C_b(F), x > 0.$$

Efficient estimation of $\pi$ through simulation: approximate $\pi$ via $(h_n)_{n \in \mathbb{N}}$ for $h_n \in C_b(F)$.

Obtain entire distribution by simulating a single path.

Warning: dynamics for $(\zeta, \chi^x)$ depend upon $p$ through $\tilde{m}, b$ functions.

- Must know $p$ to simulate. In multi-dimensional non-reversing setting this could be a problem.
PDE Argument Main Steps

Goal: show conditional c.d.f. $g$ of $X$ given $Z_0 = z$ satisfies a PDE when $\eta'\eta > 0$ and $f \in C^2(E)$.

Main Steps:

- Create a process out of $X$ (same as before)
  \[ X_t = (1/D_t) \int_t^\infty D_u f(Z_u) du \text{ for } t \geq 0. \]

- Note that
  \[ X_t = (1/D_t) \left( \int_0^\infty D_u f(Z_u) du - \int_0^t D_u f(Z_u) du \right)_{x_0}. \]

- For $x > 0$, create the process
  \[ Y_t^x := (1/D_t) \left( x - \int_0^t D_u f(Z_u) du \right). \]
PDE Argument Main Steps

\[ Y_t^x = (1/D_t) \left( x - \int_0^t D_u f(Z_u) du \right). \]

- On \( \{ X < x \} \): \( Y^x \to \infty \) since \( D_t \to 0 \).

- On \( \{ X > x \} \): \( Y^x \) hits 0 in finite time.

- On \( \{ X = x \} \): ??? but this has no probability of occurring (can independently be shown).

Thus, \( g(z, x) = \mathbb{P} \left[ X \leq x \mid Z_0 = z \right] \) is the probability \( Y^x \) does not hit 0 in finite time.
PDE Argument Main Steps

Dynamics of \((Z, Y^x)\):

\[
\begin{align*}
\cdot & \quad dZ_t = m(Z_t)dt + \sigma(Z_t)dW_t. \\
\cdot & \quad dY^x_t = (-f(Z_t) + Y^x_t\tilde{b}(Z_t))dt + Y^x_t(\theta'\sigma(Z_t)dW_t + \eta(Z_t)'dB_t). \\
\end{align*}
\]

\cdot Another (long) formula for \(\tilde{b}(z)\).

Generator \(L\) does not depend upon \(x\).

\cdot 1 - g(z,x) is the probability a diffusion with generator \(L\) hits 0 in finite time given that it starts at \((z,x)\).

If \(\eta'\eta > 0\) this diffusion is locally elliptic on \(F = E \times (0, \infty)\).

\cdot Feynman-Kac formula: \(g\) solves the pde \(Lg = 0\) on \(F\).

\cdot Thus \(\pi\) has a density satisfying \(\pi(z,x) = \partial_x g(z,x)p(z)\).
Conclusions

For the perpetuity $X_0 = \int_0^\infty D_u f(Z_u)du$,

- $Z$ is a stationary ergodic diffusion taking values in $E \subseteq \mathbb{R}^d$ with invariant measure $\pi$ and $f \in L^1(\pi)$.

- $D$ is a stochastic discount factor taking the form
  
  \[
  D_t = e^{-\int_0^t a(Z_u)du} \mathcal{E} \left( -\int_0^t \theta' \sigma(Z_u)dW_u - \int_0^t \eta(Z_u)'dB_u \right)_t.
  \]

We construct an ergodic process $(\zeta, \chi)$ with limiting distribution equal to $\pi$, the joint distribution of $(Z_0, X_0)$.

If $\eta'\eta > 0$ and $f$ is sufficiently regular, we show that $\pi$ admits a density by proving the conditional c.d.f. of $X_0$ given $Z_0 = z$ is the non-explosion probability of a locally elliptic diffusion.
Conclusions

The process \((\zeta, \chi)\) enables efficient simulation of \(\pi\) through the strong law of large numbers for ergodic processes.

The dynamics of \((\zeta, \chi)\) depend upon \(p\) and hence the simulation method is most appropriate when \(Z\) is one-dimensional or reversing.

If \(p\) is not known a work-around is the following:

- Let \(T > 0\) be “large”.
- Starting \(Z_0 = z\) run \(Z\) until \(T\).
- With \(\zeta_0 = Z_T \approx p\) and \(\chi_0 = x > 0\), run \((\zeta, \chi)\) until \(T\).
- Approximate \(\pi\) with
  \[ \int_F h \, d\pi \approx \frac{1}{T} \int_0^T h(\zeta_t, \chi_t) \, dt. \]
- Future Work: analyze the performance of this method.


