Optimal market making strategies under inventory constraints

Etienne CHEVALIER
Université d'Evry

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Joint work with:
M’hamed Gaigi, ENIT, Tunis
Vathana Ly Vath, ENSIIE & Université d'Evry
Mohamed Mnif, ENIT, Tunis
Motivations : Market making under constraints

- **Liquidity takers** :
  - trade only through market order
  - pay liquidity costs

- **Liquidity takers and providers** :
  - trade in a limit order book through market and limit order
  - pay less liquidity costs but have some inventory risk.

- **Market makers** :
  - trade in a dealer market as a single or representative market maker
  - face liquidity and inventory constraints.
Motivations : Liquidity costs for price takers

- Liquidity costs for price takers

  - Transaction costs due to bid-ask spread:

    → Shreve and Soner (1994); Korn (1998); Framstad, Oksendal and Sulem (2001),...

  - Price impact for large trades: Almgren and Chriss (2001)

    → Supply curves: Cetin, Jarrow, Protter (2004); Alfonsi, Fruth and Schied (2010),...

    → Impact functions: Bank and Baum (2004); Ly Vath, Mnif and Pham (2007); Kharroubi, Pham (2010); Roch (2011)...
Motivations: Liquidity in limit order book market

- Use limit orders instead of market orders.
  - Liquidation problems:
    - Guéant, Lehalle and Tapia (2011); Bayraktar and Ludkovski (2012); Bouchard, Lehalle and Dang (2011)
  - Market making/Portfolio management problems:
    - Avellaneda and Stoikov (2008); Guéant, Lehalle, and Tapia (2012); Guilbaud and Pham (2013)
Motivations: Market making under constraints

- A market maker in a dealer market faces some constraints
  - Provide liquidity
  - Set "reasonable" prices and spread
  - Cash and stock holdings constraints

- Ho, Stoll (1981); Huang, Simchi-Levi and Song (2012)
1 Model and problem formulation
   - Model
     - An optimal control problem with regime switching

2 Analytical properties and dynamic programming principle
   - Properties of the value functions
   - Dynamic programming principle

3 Viscosity characterization of the objective function

4 Numerical illustrations
Market making strategies

- We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

- When the $i$th buying (resp. selling) order arrives at the $\mathbb{F}$-stopping time $\theta^a_i$ (resp. $\theta^b_i$):
  - **Provide liquidity**: The market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by $P^a$ (resp. $P^b$).
  - **Set Bid and Ask prices**: The market maker may either keep the bid and ask prices constant or increase (resp. decrease) one or both of them by one tick ($\delta$).

- We consider a control $\alpha := (\epsilon^a_t, \epsilon^b_t, \eta^a_t, \eta^b_t)_{0 \leq t \leq T}$ $\mathbb{F}$-predictable process where the random variables $\epsilon^a_t, \epsilon^b_t, \eta^a_t, \eta^b_t$ are valued in $\{0, 1\}$. 
Model and problem formulation

Analytical properties and dynamic programming principle

Viscosity characterization of the objective function

Numerical illustrations

Representation of a market making strategies

Price

Time

θ^a_1  θ^a_2  θ^b_1  θ^b_2  θ^a_3  θ^b_3

Ask Price

Bid Price

Optimal market making strategies
Prices and spread dynamics

- **Bid and Ask processes**: For $c \in \{a, b\}$, the dynamics of $P^c$ evolves according to the following equations

  \[
  \begin{align*}
  \frac{dP^c_t}{P^c_t} &= 0 \quad \text{for } \xi_i < t < \xi_{i+1} \\
  P^c_{\theta^b_i} &= P^c_{\theta^b_i} - \delta \epsilon^c_{\theta^b_i} \\
  P^c_{\theta^a_i} &= P^c_{\theta^a_i} + \delta \eta^c_{\theta^a_i},
  \end{align*}
  \]
  
  for $i \in \mathbb{N}^*$, where $(\xi_i)_{i \geq 0}$ is the sequence of transaction times.

- **Mid price and spread processes**: We set $P := \frac{P^a + P^b}{2}$ and $S := P^a - P^b$. For all $i \in \mathbb{N}^*$, the dynamics of the process $(P, S)$ is given by

  \[
  \begin{align*}
  dP_t &= 0, \quad \text{for } \xi_i < t < \xi_{i+1} \\
  P_{\theta^b_i} &= P_{\theta^b_i} - \delta (\epsilon^a_{\theta^b_i} + \epsilon^b_{\theta^b_i}) \\
  P_{\theta^a_i} &= P_{\theta^a_i} + \delta (\eta^a_{\theta^a_i} + \eta^b_{\theta^a_i}) \quad \text{and} \\
  dS_t &= 0, \quad \text{for } \xi_i < t < \xi_{i+1} \\
  S_{\theta^b_i} &= S_{\theta^b_i} - \delta (\epsilon^a_{\theta^b_i} - \epsilon^b_{\theta^b_i}) \\
  S_{\theta^a_i} &= S_{\theta^a_i} + \delta (\eta^a_{\theta^a_i} - \eta^b_{\theta^a_i}).
  \end{align*}
  \]
Cash and stock holdings dynamics

**Cash holdings**: We denote by $r > 0$ the instantaneous interest rate. The bank account evolves according to the following equations

\[
\begin{align*}
\text{for } i \in \mathbb{N}^*, \quad \left\{ 
\begin{array}{l}
\frac{dX_t}{dt} = rX_t dt, \\
X_{\theta_i^b} = X_{\theta_i^b-} - P_{\theta_i^b-} \\
X_{\theta_i^a} = X_{\theta_i^a-} + P_{\theta_i^a-},
\end{array}
\right.
\end{align*}
\]

for $\xi_i < t < \xi_{i+1}$.

**Stock holdings**: The number of shares held by the market maker at time $t \in [0, T]$ is denoted by $Y_t$, and evolves according to the following equations

\[
\begin{align*}
\text{for } i \in \mathbb{N}^*, \quad \left\{ 
\begin{array}{l}
\frac{dY_t}{dt} = 0, \\
Y_{\theta_i^b} = Y_{\theta_i^b-} + 1 \\
Y_{\theta_i^a} = Y_{\theta_i^a-} - 1
\end{array}
\right.
\end{align*}
\]

for $\xi_i < t < \xi_{i+1}$.
Liquidity regimes:
Let $I$ be a continuous time, time homogeneous, irreducible Markov chain with $m$ states. The generator of the chain $I$ under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=1,...,m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain $L$ from state $i$ to state $j$.

Market orders arrivals: Let two Cox processes $N^a$ and $N^b$. The intensity processes associated with $N^a$ and $N^b$ are defined, for $t \geq 0$, by $\lambda^a(I_t, P_t, S_t)$ and $\lambda^b(I_t, P_t, S_t)$ where $\lambda^a$ and $\lambda^b$ are positive deterministic functions, bounded and defined on $\{1, ..., m\} \times \frac{\delta}{2} \mathbb{N} \times \delta \mathbb{N}$.

We define $\theta_k^a$ (resp. $\theta_k^b$) as the $k^{th}$ jump time of $N^a$ (resp. $N^b$), which corresponds to the $k^{th}$ buy (resp. sell) market order.
Admissible strategies

- **Liquidity constraints**: Let $K > 0$, the market maker has to use controls such that

$$P_t - S_t/2 > 0 \quad \text{and} \quad 0 < S_t \leq K \times \delta, \quad \text{for } 0 \leq t \leq T.$$

- **Inventory and cash constraints**: Let $x_{min} < 0$ and $y_{min} \leq y_{max}$. We introduce the following notations:

$$S = (x_{min}, +\infty) \times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2} \mathbb{N} \times \delta\{1, ..., K\},$$

$$S = \{(t, x, y, p, s) \in [0, T] \times S : p - \frac{s}{2} \geq \delta\}.$$

For a control $\alpha$, we define the liquidation time:

$$\tau^{t, i, z, \alpha} := \inf\{u \geq t : X_u^{t, i, x, \alpha} \leq x_{min} \text{ or } Y_u^{t, i, y, \alpha} \in \{y_{min} - 1, y_{max} + 1\}\}$$

- **Admissible strategies**: Let $(t, z) := (t, x, y, p, s) \in S$, the strategy $\alpha = (\epsilon_u^a, \epsilon_u^b, \eta_u^a, \eta_u^b)_{t \leq u \leq T}$ is admissible, if the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{0, 1\}$ and for all $u \in [t, T]$, $(u, Z_u^{t, i, z, \alpha}) \in S$. We denote by $A(t, z)$ the set of all these admissible policies.
Objective function

- **Portfolio liquidation**: If the market maker decides (or has) to liquidate her portfolio, then she actually gets
  \[ Q(t, y, p, s) = (p - \text{sign}(y)\frac{S}{2})f(t, y), \]
  where \( f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+ \), non-linear in \( y \) and such that
  \[ f(t, y) \leq f(t, y') \text{ if } y' \leq y \text{ and } yf(t', y) \leq yf(t, y) \text{ if } t' \leq t. \]

- **Utility and penalty functions**: Let \( \gamma > 0 \) and \( U(x) = 1 - e^{-\gamma x} \) on \( \mathbb{R} \). We set
  \[ U_L = U \circ L \quad \text{where } L(t, x, y, p, s) = x + yQ(t, y, p, s). \]
  Let \( g \) a bounded positive function defined on \( \{y_{\text{min}}, \ldots, y_{\text{max}}\} \).

- **Objective function**: We consider the functions \((v_i)_{i \in \{1, \ldots, m\}}\) defined on \( S \) by
  \[ v_i(t, z) := \sup_{\alpha \in A(t, z)} J_i^\alpha(t, z) \]
  where we have set
  \[ J_i^\alpha(t, z) := \mathbb{E} \left[ U_L(T \land \tau^{t, i, z, \alpha}, Z^{t, i, z, \alpha}_{(T \land \tau^{t, i, z, \alpha})_+}) \right] - \int_t^{T \land \tau^{t, i, z, \alpha}} g(Y^{t, i, y, \alpha}_s) ds. \]
Analytical properties and dynamic programming principle

- Model and problem formulation
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- Numerical illustrations
Let \((t, z) := (t, x, y, p, s) \in S\). From monotonicity of \(f\),

\[
L(t, z) \geq x_{\text{min}} + y_{\text{min}} f(0, y_{\text{min}})(p - \frac{K\delta}{2}).
\]

**Proposition**

There exist \(C_1, C_2\) and \(C_3\) positive constants such that

\[
1 - C_1 - C_2 e^{C_3 p} \leq v_i(t, z) \leq 1, \quad \forall (i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times S,
\]
Uniform continuity of the objective functions

Hölder continuity of the criteria functions

Let $i \in \{1, \ldots, m\}$, $(t, z) := (t, x, y, p, s) \in \bar{S}$ and $(t', x')$ in $[0, T] \times (x_{min}, +\infty)$. For all $\alpha \in A(t \wedge t', z)$ such that $\alpha\|_{[t \wedge t', t \vee t']} = 0$, we have $\alpha \in A(t, z) \cap A(t', z')$ with $z' = (x', y, p, s)$ and, if $(t^{\prime \prime}, x')$ is close enough to $(t, x)$, then

$$| J_i^\alpha(t, z) - J_i^\alpha(t', z') | \leq K_2(p) \left( \psi(r e^r T \mid x'(t - t') \mid) + \psi(x' - x) + | t' - t | \right).$$

where $K_2(p) > 0$ and $\psi$ an Hölder continuous function on $\mathbb{R}$.

Uniform continuity of the objective functions

Let $(i, y, p, s) \in \{1, \ldots, m\} \times \{y_{min}, \ldots, y_{max}\} \times \delta^2 \mathbb{N}^* \times \delta \{1, \ldots, K\}$ such that $p - \frac{s}{2} > 0$.

The function $(t, x) \rightarrow v_i(t, x, y, p, s)$ is uniformly continuous on $[0, T] \times [x_{min}, +\infty)$.
Dynamic programming principle

Let \((i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times S\). Let \(\nu\) be a stopping time in \(\mathcal{T}_t, T\), we have

\[
v_i(t, z) = \sup_{\alpha \in A(t, z)} \mathbb{E}\left[ v_{i, \nu \wedge \hat{\theta}}(\nu \wedge \hat{\theta}, Z_{\nu \wedge \hat{\theta}}^{t, i, z, \alpha}) \mathbb{1}_{\{\nu \wedge \hat{\theta} < \hat{T}_\alpha\}} + U_L \left( \hat{T}_\alpha, x \exp(\hat{T}_\alpha - t), y, p, s \right) \mathbb{1}_{\{\hat{T}_\alpha \leq \nu \wedge \hat{\theta}\}} - g(y) \left( \nu \wedge \hat{\theta} \wedge \hat{T}_\alpha - t \right) \right],
\]

with \(\hat{T}_\alpha = t, i, z, \alpha \wedge T\) and

\[
\hat{\theta} = \inf\{u \geq t : N_u > N_{u-} \text{ or } N_{u}^{a, i, t, z} > N_{u-}^{a, i, t, z} \text{ or } N_{u}^{b, i, t, z} > N_{u-}^{b, i, t, z} \}.
\]
Analytical properties of the objective function and dynamic programming principle

- Model and problem formulation
- Analytical properties and dynamic programming principle
- \textit{Viscosity characterization of the objective function}
- Numerical illustrations
HJB equation (1)

- **Set of admissible controls**: We define the following set:
  \[
  A(t, z) := \{ \alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{0, 1\}^4 : \delta \varepsilon^b < p - \frac{s}{2},
  \delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta \text{ and } \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta \}.
  \]

- **Transactions operators**: For all \((i, t, x, y, p, s) := (i, t, z) \in \{1, ..., m\} \times S\) and \(\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b\} \in A(t, z)\), we introduce the two operators:
  \[
  \mathcal{A} v_i(t, z, \alpha) = \begin{cases} 
  U_L(t, x, y_{\min}, p, s) & \text{if } y = y_{\min}, \\
  v_i(t, x + p + \frac{s}{2}, y - 1, p + \frac{\delta}{2}(\eta^a + \eta^b), s + \delta(\eta^a - \eta^b)) & \text{else.}
  \end{cases}
  \]
  \[
  \mathcal{B} v_i(t, z, \alpha) = \begin{cases} 
  U_L(t, x, y_{\max}, p, s), & \text{if } y = y_{\max} \\
  U_L(t, z) & \text{if } x < x_{\min} + p - \frac{s}{2} \text{ or } x = x_{\min} + p - \frac{s}{2} < 0 \\
  v_i(t, x - p + \frac{s}{2}, y + 1, p - \frac{\delta}{2}(\varepsilon^a + \varepsilon^b), s - \delta(\varepsilon^a - \varepsilon^b)) & \text{else.}
  \end{cases}
  \]
Let $(\varphi_i)_{1 \leq i \leq m}$ a family of smooth functions defined on $S$. We introduce the following operator associated with state $i \in \{1, .., m\}$ :

$$
\mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) = \begin{cases} 
rx \frac{\partial \varphi_i}{\partial x} + \sum_{j \neq i} \gamma_{ij} (\varphi_j(t, x, y, p, s) - \varphi_i(t, x, y, p, s)) - g(y) \\
+ \sup_{\alpha \in \mathcal{A}(t, z)} \left[ \lambda^a_i (p, s) (A \varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)) + \lambda^b_i (p, s) (B \varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)) \right] = 0.
\end{cases}
$$

We consider the HJB equation :

$$
- \frac{\partial \varphi_i}{\partial t} - \mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) = 0, \quad \text{for} \ (t, z) \in S, \quad (1)
$$

with the following boundary and terminal conditions :

$$
v_i(t, x_{\text{min}}, y, p, s) = U_L(t, x_{\text{min}}, y, p, s) \quad (2)
$$

$$
v_i(T, x, y, p, s) = U_L(T, x, y, p, s) \quad (3)
$$
Theorem:

The family of objective functions \((v_i)_{1 \leq i \leq m}\) is the unique family of functions such that

i) **Continuity condition**: For all \((i, y, p, s) \in \{1, \ldots, m\} \times \{y_{\text{min}}, \ldots, y_{\text{max}}\} \times \frac{\delta}{2} \mathbb{N} \times \delta\{1, \ldots, K\}\), \((t, x) \to v_i(t, x, y, p, s)\) is continuous on \(\{(t, x) \in [0, T) \times [x_{\text{min}}, +\infty) : (t, x, y, p, s) \in S\}\).

ii) **Growth condition**: There exists \(C_1, C_2\) and \(C_3\) positive constants such that

\[
1 - C_1 - C_2 e^{C_3 p} \leq v_i(t, x, y, p, s) \leq 1, \quad \text{on} \{1, \ldots, m\} \times S.
\]

iii) **Boundary conditions**:

\[
v_i(t, x_{\text{min}}, y, p, s) = U_L(t, x_{\text{min}}, y, p, s) \quad \text{and} \quad v_i(T, x, y, p, s) = U_L(T, x, y, p, s).
\]

iv) **Viscosity solution**: \((v_i)_{1 \leq i \leq m}\) is a viscosity solution of the system of variational inequalities (1) on \(\{1, \ldots, m\} \times S\).
Numerical illustrations

- Model and problem formulation
- Analytical properties and dynamic programming principle
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Numerical values

- Market values:
  → Initial conditions: $x = 5$, $y = 2$, $p = 1$, $s = 0.02$.
  → $r = 0.05$, $\delta = 0.02$, $\lambda = 20$.
  → Impact function: $f(t, y) = \exp(0.09y(T - t))$.
  → Intensity functions:

$$\lambda_i^a(p, s) = \frac{\psi_i^a}{p} \exp(-s-0.01(p-1)) \quad \text{and} \quad \lambda_i^b(p, s) = \psi_i^b p \exp(-s+0.01(p-1)),$$

with $\psi_1^a = 120$, $\psi_2^a = 80$, $\psi_1^b = 80$, $\psi_2^b = 120$.

- Constraints:
  → $x_{\min} = -20$, $y_{\min} = -10$, $y_{\max} = 10$, $K = 5$, $T = 1$.
  → Penalty function: $g(y) = y^2 \times 10^{-3}$.
  → Utility function: $U(l) = 1 - e^{-0.01l}$ i.e. $\gamma = 0.01$.

- Numerical values:
  → Localisation: $x_{\max} = 20$, $p_{\min} = 1 - 20 \times \frac{\delta}{2}$, $p_{\max} = 1 + 20 \times \frac{\delta}{2}$
  → Discretization: $n_x = 40$ and $n_t = 20$. 
A cash holdings path

**Figure**: A cash holdings path
A stock holdings path

**Figure:** A stock holdings path
Figure: Bid and ask price paths
**Introduction**

**Model and problem formulation**

**Analytical properties and dynamic programming principle**

**Viscosity characterization of the objective function**

**Numerical illustrations**

**Liquidation value**

**Figure:** A path of $L(t, Z_t)$