Estimates for Trajectories Confined to a Cone in $\mathbb{R}^n$

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Abstract

We develop estimates on the distance of a trajectory, associated with a differential inclusion $\dot{x} \in F$ in $\mathbb{R}^n$, to a given set of feasible $F$-trajectories, where ‘feasible’ means ‘with values confined to a given cone’. When the cone is a half space, it is known that a (linear) $K h(x)$ estimate of the $W^{1,1}$ distance is valid, where $K$ is a constant independent of the initial choice of trajectory and $h(x)$ is a measure of the constraint violation. A recent counter-example has unexpectedly demonstrated that linear estimates of the distance are no longer valid when the state constraint set is the intersection of two half spaces. This paper addresses fundamental questions concerning the approximation of general $F$-trajectories by $F$-trajectories confined to a cone which is the intersection of two half-spaces. In this context, we establish the validity of a (super-linear) $K h(x) |\ln h(x)|$ estimate of the distance. We demonstrate by means of an example that the structure of this estimate is optimal. We show furthermore that a linear estimate can be recovered in the case when the velocity set $F$ is strictly convex.

Keywords: differential inclusions, state constraints, Filippov theorems.

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1 Introduction

Consider a control system, formulated as a differential inclusion:

$$\dot{x}(t) \in F(t, x(t)) \quad \text{for a.e. } t \in [0, 1],$$

(1)

with an associated state constraint

$$x(t) \in A(t) \quad \text{for all } t \in [0, 1],$$

for which the data comprise: an integer \( n \geq 1 \) and multifunctions \( F : \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) and \( A : \mathbb{R} \rightrightarrows \mathbb{R}^n \). It is assumed that the state constraint sets \( A(t) \) can be expressed in terms of the smooth functions \( h_j : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, j = 1, \ldots, r \), for some integer \( r \), as follows:

$$A(t) = \bigcap_{j=1}^{r} \left\{ x \in \mathbb{R}^n \mid h_j(t, x) \leq 0 \right\} \quad \text{for all } t \in [0, 1].$$

(2)

An absolutely continuous \( \mathbb{R}^n \)-valued function \( x(\cdot) \), defined on some interval \([t_1, t_2] \subseteq [0, 1]\), which satisfies the differential inclusion \( \dot{x}(t) \in F(t, x(t)) \) for a.e. \( t \in [t_1, t_2] \) will be called an \( F \)-trajectory. We will say that \( x(\cdot) \) is a feasible \( F \)-trajectory if, additionally, it satisfies \( x(t) \in A(t) \) for all \( t \in [t_1, t_2] \). Given an \( F \)-trajectory \( x(\cdot) \) on \([0, 1]\), we write

$$h(x) = \max_{t \in [0,1]} \left\{ h_1(t, x(t)) \lor \ldots \lor h_r(t, x(t)) \lor 0 \right\}.$$

The number \( h(x) \) quantifies the extent to which the trajectory \( x(\cdot) \) violates the state constraint.

This paper concerns estimates on the distance of a given \( F \)-trajectory \( \hat{x}(\cdot) \) from the set of feasible \( F \)-trajectories in terms of the state constraint violation \( h(\hat{x}) \), of the type:

$$\inf \left\{ \| \dot{x}(\cdot) - x(\cdot) \|_{\mathcal{W}^{1,1}} \mid x(\cdot) \in \mathcal{F} \right\} \leq K \cdot \theta(h(\hat{x})).$$

Here, \( \| \cdot \|_{\mathcal{W}^{1,1}} \) is the norm

$$\| z(\cdot) \|_{\mathcal{W}^{1,1}} = \| z(0) \| + \| \dot{z}(\cdot) \|_{L^1},$$

\( \theta : [0, \infty) \rightarrow [0, \infty) \) is some continuous, increasing function satisfying \( \theta(0) = 0 \) (a “modulus” of the state constraint violation), and \( K \) is a constant, independent of \( \hat{x}(\cdot) \). The infimum is taken over the set \( \mathcal{F} \) of feasible \( F \)-trajectories \( x(\cdot) \) satisfying a specified initial condition.

Special cases of this inequality, in which \( \theta(h(\hat{x})) = h(\hat{x}) \) (a ‘linear’ estimate of the distance), or weaker versions in which the \( \mathcal{W}^{1,1} \) norm is replaced by the \( L^\infty \) norm, have received considerable attention in the literature, because of their role in the derivation of non-degenerate forms of the state-constrained Maximum Principle for \( \mathcal{W}^{1,1} \) local minimizers of optimal control problems, in related sensitivity analysis and other areas of state constrained system theory and optimal control. A partial list of relevant references is the following: [14], [9], [11] and [7] where linear estimates in the \( L^\infty \) norm have been obtained under an ‘inward pointing hypothesis; [3] in which a proof where one state constraint and \( \mathcal{W}^{1,1} \) norm are employed; [1], [2] where a non-anticipative Filippov type theorem has been investigated with applications to characterization of the value function in Differential Games and Optimal Control theory; finally see [4], [8], [10] and [12] for the applications to state constrained Maximum Principle, dynamic programming for optimal control problems and derivation of sensitivity relations.

But super-linear estimates are also of interest, because they may be used, for example, to characterize the value function of state constrained optimal control problems in terms of a solutions to the Hamilton-Jacobi equation (appropriately defined) and to investigate its regularity properties.
It is known [3] that a linear $W^{1,1}$ estimate
\[
\inf \| \hat{x}(\cdot) - x(\cdot) \|_{W^{1,1}} \leq K \cdot h(\hat{x}).
\]
is valid, in the case of one state constraint (i.e., $r = 1$), under ‘standard’ linear growth, measurable Lipschitz conditions on $F$, and under an ‘inward pointing’ hypothesis concerning the existence of a velocity achieving strict decrease of all the active state constraint functionals.

It is also known that linear $W^{1,1}$ estimates are not in general valid for multiple state constraints. This latter assertion was recently confirmed in [3] by means of a counter-example, involving a convex velocity set $F$ independent of $t$ and $x$ and two state constraint functionals linear in $x$ and independent of $t$.

The counter-example raises some fundamental questions regarding the approximation of general $F$-trajectories by the sub-class of $F$-trajectories confined to a cone:

(a): When we pass from a single state constraint to a multiple state constraint setting is it possible to establish a weaker estimates in terms of the constraint violation, replacing the linear modulus that is no longer valid?

(b): Is it possible to identify significant classes of state constrained control systems involving multiple state constraints for which linear $W^{1,1}$ estimates are still valid?

We confine attention to the case where $F$ is independent of $t$ and $x$, and where $h_1$ and $h_2$ are linear functions of $x$, independent of $t$ (the ‘constant $F$, $A$ the intersection of two half spaces’ case). Henceforth then we take $F$ and $A$ to be subsets of $\mathbb{R}^n$ and study the properties of the state constrained control system:
\[
\dot{x}(t) \in F \text{ a.e. } t \in [0, 1],
x(t) \in A \text{ for all } t \in [0, 1],
\]
when $A$ has representation
\[
A = \{ x \in \mathbb{R}^n \mid b_1 \cdot x \leq 0 \text{ and } b_2 \cdot x \leq 0 \}
\]
for some linearly independent vectors $b_1, b_2 \in \mathbb{R}^n$.

We provide positive answers to the earlier questions, at least in the ‘constant $F$, $A$ is the intersection of two half spaces’ case. Concerning (a), we establish estimates involving the modulus of state constraint violation
\[
\theta(h) = h | \ln(h) |, \quad (0 < h \leq 1/4)
\]
and we exploit this ‘super-linear’ estimate to prove new regularity properties of a value function for a state constrained optimal control problem. We show furthermore that the $h | \ln(h) |$ structure of these estimates is optimal. Concerning (b), we identify situations where linear estimates are valid; these include the important case when the velocity set $F$ is strictly convex. It is noteworthy however that, even in the ‘constant $F$, $A$ is the intersection of two half spaces’ setting, the fundamental differences are revealed between the types of distance estimates that can be derived depending on whether the state constraint has a smooth boundary or a boundary with ‘edges’.

Besides identifying a new structure for distance estimates based on the $h | \ln(h) |$ modulus appropriate to control systems with multiple state constraints, and developing methodologies for establishing such estimates, our work has other novel features. In earlier related research the existence of an inward pointing vector belonging to $F$ is typically hypothesized; here we
assume only the existence of an inward pointing vector belonging to the convex hull of \( F \). We also provide estimates, for the first time, involving a distance function on \( F \)-trajectories related to the Ekeland metric, which is strictly stronger than the \( W^{1,1} \) distance.

Addendum. At the request of the Editor we briefly review research developments since submission of this paper for publication, within a more general setting in which the velocity set \( F(x) \) depends on \( x \) and \( A \) is a closed set, possibly not of the form (4). Assume that \( F(\cdot) \) is a compact valued multifunction, which has linear growth and is locally Lipschitz continuous. Assume furthermore that an appropriate ‘inward pointing’ hypothesis is satisfied (the precise form of which will depend on the manner in which the state constraint is described).

(i): If \( A \) compact convex, it is possible to derive an \( h|\ln(h)| \) estimate, by means of quite different analytical techniques to those employed in this paper, see [6], where \( h \) is interpreted in terms of the Euclidean distance to \( A \).

(ii): If \( F \) is strictly convex and \( A \) can be represented as the intersection of a finite number of \( W^{1,\infty} \) functional inequality constraints, only at most two of which are active at any point in the state space, then a generalization of the techniques employed in this paper yields a linear estimate. (A version of these results is announced in [5].)

(iii): if we further broaden the framework and allow \( F(t, x) \) to depend on \( t \) as well as \( x \) then, even if \( F(t, x) \) is continuous in \( t \) (and Lipschitz in \( x \)), then the \( h|\ln h| \) may fail. (see [5].)

Notation: given a convex set \( C \subset \mathbb{R}^n \) we denote by \( \pi_C(y) \) the (unique) projection of the point \( y \) into \( C \). The closed unit ball in \( \mathbb{R}^n \) is denoted by \( \mathbb{B} \). Given a set \( Y \subset \mathbb{R}^n \), \( \text{co} \ Y \) denotes the convex hull of \( Y \).

2 Distance Estimates

Consider the state constrained control system (3) in which \( F \) is a compact set in \( \mathbb{R}^n \) and \( A \) has representation (4) as the intersection of two closed half-spaces in \( \mathbb{R}^n \) with linearly independent normal vectors \( b_1 \) and \( b_2 \).

Given \([t_0, t_1] \subset [0, 1]\) and a function \( x : [t_0, t_1] \to \mathbb{R}^n \) we write

\[
h(x) = \max_{t \in [t_0, t_1]} \left\{ b_1 \cdot x(t) \lor b_2 \cdot x(t) \lor 0 \right\}.
\]

Moreover, given measurable functions \( u, v : [t_0, t_1] \to \mathbb{R}^n \) we define their ‘Ekeland distance’ as

\[
d_{[t_0,t_1]}(u,v) = \text{meas}\left\{ t \in [t_0, t_1] ; \ u(t) \neq v(t) \right\}.
\]

Theorem 2.1 below provides estimates on the distance of a given \( F \)-trajectory \( \hat{x}(\cdot) \) to the set of \( F \)-trajectories satisfying the state constraint and having a specified initial value in \( A \), in terms of the state constraint violation \( h(\hat{x}) \). As \( h \downarrow 0 \), these estimates provide an \( O(1) \cdot h|\ln h| \) bound on the distance, in general. If however the velocity set \( F \) satisfies a strict convexity condition, then this super-linear estimate can be replaced by a stronger, linear estimate.

**Theorem 2.1**. Consider a compact set \( F \subset \mathbb{R}^n \) and a set \( A \subset \mathbb{R}^n \) with representation (4) where \( b_1 \) and \( b_2 \) are linearly independent vectors. Fix \( K_0 \geq 0 \) and assume

\[(H1)\quad \text{co} \ F \cap \text{int} \ A \neq \emptyset \; .\]
Then there exists $K \geq 0$ with the following property: Given any $t_0 \in [0, 1]$, $F$-trajectory $\hat{x} : [t_0, 1] \to \mathbb{R}^n$ satisfying $h(\hat{x}) \leq \frac{1}{4}$ and $x_0 \in A$ satisfying $|\hat{x}(t_0) - x_0| \leq K_0 h(\hat{x})$, there exists an $F$-trajectory $x : [t_0, 1] \to \mathbb{R}^n$ such that $x(t_0) = x_0$, $x(t) \in A$ for all $t \in [t_0, 1]$ and

$$d_{[t_0, 1]}(\hat{x}, \dot{x}) \leq K h(\hat{x}) |\ln(h(\hat{x}))| . \quad (5)$$

If it is additionally assumed that

\textbf{(H2)}: $\pi_{\text{span}\{b_1, b_2\}} F$ is strictly convex

then, possibly following an adjustment of the constant $K$, we have

$$d_{[t_0, 1]}(\hat{x}, \dot{x}) \leq K h(\dot{x}) . \quad (6)$$

The theorem is proved in Section 4 below.

Comments

(i): Since, in the statement of the theorem, the initial state $x_0$ is required to satisfy $|\hat{x}(t_0) - x_0| \leq K_0 h(\hat{x})$, by possibly increasing $K$ we can arrange so that

$$E_{[t_0, 1]}(x, \dot{x}) \leq K h(\dot{x}) |\ln(h(\dot{x}))| .$$

where

$$E_{[t_0, 1]}(x, \dot{x}) \doteq |x(t_0) - \hat{x}(t_0)| + \text{meas} \{t \in [t_0, 1] | \dot{x}(t) \neq \hat{x}(t)\} .$$

Since the velocity set $F$ is compact, the metric $E_{[t_0, 1]}(x, \dot{x})$ is stronger than the $\|x - \hat{x}\|_W^{1, 1}(t_0, 1)$ metric. Therefore, Theorem 2.1 remains valid if we replace $d_{[t_0, 1]}(\hat{x}, \dot{x})$ by the distance $\|x - \hat{x}\|_W^{1, 1}(t_0, 1)$. We expect that the sharper $E_{[t_0, 1]}(x, \dot{x})$ estimate will be useful in establishing regularity properties of the value function for optimal control problems involving a cost integrand discontinuous in the state velocity.

(ii): Notice that hypothesis (H1) requires the existence of a vector in the interior of the tangent cone to the state constraint set, which lies merely in the convex hull of the velocity set $F$. This is a significant weakening of the ‘inward pointing’ hypotheses employed elsewhere (for purposes of establishing distance estimates), all of which apparently require the vector to lie in the set $F$ itself.

(iii): In [3] an example of the control system (3) with $n = 2$ was studied, in which

$$F = \text{co} \left\{ (2, 1), (-2, 1), (0, 0) \right\} \quad \text{and} \quad b_1 = (1, -1), b_2 = (-1, -1) .$$

The authors constructed a family of $F$-trajectories $\{x_\epsilon : [0, 1] \to \mathbb{R}^2 | \epsilon > 0\}$ satisfying $x_\epsilon(0) = (0, 0)$ and $h(x_\epsilon) = \epsilon$ for all $\epsilon$ with the following property: there exists a constant $c > 0$ such that, for each $\epsilon > 0$ sufficiently small we have

$$\|x_\epsilon - \hat{x}\|_W^{1, 1}(0, 1) \geq c h(x_\epsilon) |\ln(h(x_\epsilon))|$$

for every feasible trajectory $x(\cdot)$ with $x(0) = (0, 0)$. This shows that the estimate structure (5) is optimal, because in general the conclusions are no longer valid if $h |\ln h|$ is replaced by any modulus $\theta(h)$ approaching zero more rapidly as $h \downarrow 0$. Notice that in this example the projection of the velocity set $\pi_{\text{span}\{b_1, b_2\}} F$, which coincides with $F$ itself, is not strictly convex. The example also illustrates that, in order for a linear estimate to be valid, one cannot entirely remove the strict convexity assumption.
(iv): Metric regularity of a mapping defining a constraint is an important, and much studied, property in optimization, yielding information about stability of solution sets, non-degeneracy of Lagrange multiplier rules etc. We briefly comment on the links with Thm. 2.1. Fix $x_0 \in A$, $t_0 \in [0, 1]$ and define the set of $F$-trajectory’s on $[t_0, 1]$ with initial value $x(t_0) = x_0$ to be

$$S(t_0, x_0) := \{ x(\cdot) : [t_0, 1] \to \mathbb{R}^n \ \text{F-trajectory} \},$$

and assume that the hypotheses (H1) and (H2) are in force. Now consider the set valued mapping $H : S(t_0, x_0) \to \mathbb{R}^+$:

$$H(x(\cdot)) = [h(x), +\infty).$$

This is a possible choice of mapping, for studying state-constrained differential inclusions, within the framework of metric regularity. Notice that the inverse mapping is

$$H^{-1}(h) = \{ x(\cdot) \in S(t_0, x_0) \mid h(x) \leq h \}.$$

We readily deduce from Thm. 2.1 that the inverse mapping is Lipschitz continuous (with the $W^{1,1}$ metric on $S(t_0, x_0)$) and therefore epi-Lipschitz at $(\bar{x}(\cdot), \bar{h})$, where $\bar{x}(\cdot)$ is an arbitrary point in $S(t_0, x_0)$ and $\bar{h} = h(\bar{x})$. But epi-Lipschitz continuity of the inverse mapping is a characterization of metric regularity (see for instance [13, Thm. 9.43]). Thus Thm. 2.1 may be interpreted as providing sufficient conditions for metric regularity of a mapping associated with the state constraint $x(t) \in A$.

### 3 An Application: Regularity Properties of the Value Function

Take compact sets $F$ and $A$ in $\mathbb{R}^n$. Assume $A$ has representation (4) in terms of linearly independent vectors $b_1$ and $b_2$. Consider a family of optimization problems, parameterized by the initial time and state $(\tau, \xi) \in [0, 1] \times A$:

$$P(\tau, \xi) \\begin{cases} \text{Minimise } J(\tau, \xi) = \int_{\tau}^{\tau+1} L(t, x(t), \dot{x}(t)) \, dt \\ \text{over } F\text{-trajectories } x(\cdot) \text{ on } [\tau, 1] \text{ satisfying} \\ x(t) \in A \text{ for all } t \in [\tau, 1] \\ x(\tau) = \xi. \end{cases}$$

We shall invoke the hypothesis

(C1): $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given function such that

(i) $L(t, y, w)$ is locally bounded;

(ii) $L(\cdot, y, w)$ is measurable for all $(y, w) \in \mathbb{R}^n \times \mathbb{R}^n$;

(iii) $L(t, \cdot, \cdot)$ is locally Lipschitz uniformly w.r.t. $t \in [0, 1]$.

We define the value function $V : [0, 1] \times A \to \mathbb{R}$ as follows. For any $(\tau, \xi) \in [0, 1] \times A$, $V(\tau, \xi)$ is the infimum cost of $P(\tau, \xi)$. (Under the hypotheses we shall impose, there exist feasible $F$-trajectories for all initial data $(\tau, \xi) \in [0, 1] \times A$, so $V(\cdot, \cdot)$ is unambiguously defined in this way.)

**Proposition 3.1** Consider a compact set $F$ and a set $A$ with representation (4) where $b_1$ and $b_2$ are linearly independent vectors. Assume that (H1) and (C1) are satisfied. Then, for any $r > 0$, there exists a positive constant $K_r$ with the following properties:
(i) 
\[ |V(z_1) - V(z_2)| \leq K_r |z_1 - z_2| \cdot |\ln(|z_1 - z_2|)| \]
for all \( z_1, z_2 \in [0, 1] \times (A \cap rB) \) such that \(|z_1 - z_2| \leq \frac{1}{4} \).

(ii) If additionally (H2) is satisfied then
\[ |V(z_1) - V(z_2)| \leq K_r |z_1 - z_2| \]
for all \( z_1, z_2 \in [0, 1] \times (A \cap rB) \).

**Proof.** We supply a proof only of (i); that of (ii) is along essentially similar lines.

Notice that, since \( F \) is compact, for any \( \tau \in [0, 1] \) and any \( F \)-trajectory \( x(\cdot) \) on \([\tau, 1]\) such that \( x(\tau) \in rB \) we have: \( x(t) \in (r + \kappa)B \) for all \( t \in [\tau, 1] \), where
\[ \kappa \doteq \max_{w \in F} |w|. \]  
By (C1) we can find two positive constants \( k_r \) and \( L_r \) such that
\[ |L(t, y, w)| \leq L_r \quad \text{for all} \quad (t, y, w) \in [0, 1] \times (r + \kappa)B \times F \]
and
\[ |L(t, y_1, w_1) - L(t, y_2, w_2)| \leq k_r(|y_1 - y_2| + |w_1 - w_2|) \quad \text{for all} \quad (y_1, w_1), (y_2, w_2) \in (r + \kappa)B \times F. \]

Fix two points \( z_1 = (\tau_1, \xi_1), z_2 = (\tau_2, \xi_2) \in [0, 1] \times \mathbb{R}^n \) as in the statement of the proposition. It is not restrictive to assume that \( \tau_1 \leq \tau_2 \). Fix any \( \varepsilon > 0 \). Then there exists a feasible \( F \)-trajectory \( x_\varepsilon(\cdot) \) such that \( x_\varepsilon(\tau_1) = \xi_1 \) and
\[ V(z_1) \geq \int_{\tau_1}^{1} L(t, x_\varepsilon(t), \dot{x_\varepsilon}(t)) \, dt - \varepsilon. \]  
Consider the point \( (\tau_2, x_\varepsilon(\tau_2)) \) and the \( F \)-trajectory \( \hat{x}(\cdot) \) on \([\tau_2, 1]\) defined by \( \hat{x}(t) \doteq x_\varepsilon(t) + \xi_2 - x_\varepsilon(\tau_2) \). Then, we obtain \( \dot{x}(\tau_2) = \xi_2 \) and
\[ \|\dot{x}(\cdot) - x_\varepsilon(\cdot)\|_{W^{1,1}(\tau_2, 1)} \doteq |\xi_2 - x_\varepsilon(\tau_2)|. \]  
Moreover, by Thm. 2.1 there exists a feasible \( F \)-trajectory \( x(\cdot) \) such that \( x(\tau_2) = \xi_2 \) and
\[ \|x(\cdot) - \hat{x}(\cdot)\|_{W^{1,1}(\tau_2, 1)} \doteq K h(\hat{x})|\ln h(\hat{x})|, \]  
for some positive constant \( K \) that does not depend on the choice of \( \varepsilon, z_1 \) or \( z_2 \). But \( h(\hat{x}) \leq \|\dot{x}(\cdot) - x_\varepsilon(\cdot)\|_{W^{1,1}(\tau_2, 1)} \) and \( |\xi_2 - x_\varepsilon(\tau_2)| \leq |\xi_1 - \xi_2| + \kappa|\tau_1 - \tau_2| \). From (9) and (10), it follows that
\[ \|x(\cdot) - x_\varepsilon(\cdot)\|_{W^{1,1}(\tau_2, 1)} \doteq K_2 |z_1 - z_2| \cdot \left[ 1 + |\ln |z_1 - z_2|| \right], \]
for some positive constant \( K_2 \) that does not depend on the choice of \( \varepsilon, z_1 \) or \( z_2 \).
Therefore we have

\[ V(z_2) - V(z_1) \leq \int_{t_2}^{t_1} L(t, x(t), \dot{x}(t)) \, dt - \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t)) \, dt + \varepsilon \]

\[ - \int_{t_2}^{t_1} L(t, x(t), \dot{x}(t)) \, dt + \varepsilon \]

\[ \leq 2k_r \|x(\cdot) - x^*(\cdot)\|_{W^{1,1}(\tau_2, \tau_1)} + L_r |\tau_2 - \tau_1| + \varepsilon \]

\[ \leq K_r |z_1 - z_2| \cdot |\ln |z_1 - z_2|| + \varepsilon , \]

where the constant \( K_r > 0 \) does not depend on \( \varepsilon, z_1 \) or \( z_2 \).

The reverse inequality

\[ V(z_1) - V(z_2) \leq K_r |z_1 - z_2| \cdot |\ln |z_1 - z_2|| + \varepsilon \]

is easily obtained by means of similar arguments. The assertions of the proposition now follow, by taking the limit as \( \varepsilon \) tends to zero.

**Example.** We now supply an example of the optimization problem \( P(\tau, \xi) \) above, involving sets \( A \subset \mathbb{R}^2, F \subset \mathbb{R}^2 \) and a cost integrand \( L \) satisfying hypotheses (H1) and (C1), which demonstrates that the structure of the regularity estimate provided by Prop. 3.1 is optimal. Define the zig-zag function \( x^*(\cdot) : [0, 1] \to \mathbb{R} \) to be the function such that

\[ x^*(0) = 0, \quad x^*(t_k) = (-1)^k \frac{t_k}{2}, \]

and

\[ \dot{x}^*(t) = (-1)^k \text{ for } (t_{k+1}, t_k), \]

where \( t_k \) is the decreasing sequence of times

\[ t_k = \frac{1}{3^k}, \quad k = 0, 1, 2, 3 \ldots \]

with \( x^* \) piecewise affine along each interval \([t_{k+1}, t_k]\). Consider the cone \( A \subset \mathbb{R}^2 \):

\[ A \doteq \left\{ (x, y) : |x| \leq y \right\} = \left\{ (x, y) : b_1 \cdot (x, y) \leq 0 \text{ and } b_2 \cdot (x, y) \leq 0 \right\}, \]

where \( b_1 = (1, -1) \) and \( b_2 = (-1, -1) \). Define also the set \( F \subset \mathbb{R}^2 \)

\[ F \doteq \text{co} \left\{ (-1, \frac{1}{2}), (1, \frac{1}{2}), (0, 0) \right\} \]

and the function

\[ L(t, y, w = (w_1, w_2)) = L(t, w_1) \doteq \left\{ \begin{array}{ll} 1 - w_1 & \text{if } \dot{x}^*(t) = 1, \\ w_1 + 1 & \text{if } \dot{x}^*(t) = -1. \end{array} \right. \quad (11) \]

Now consider the class of problems \( P(\tau, \xi), (\tau, \xi) \in [0, 1] \times A \) with these identifications of \( F \) and \( A \). Let the value function \( V : [0, 1] \times A \to \mathbb{R} \) be defined as above.

It is clear from the definition of \( L \) that \( L(t, w_1) \geq 0 \) whenever \(|w_1| \leq 1 \) and that the \( F \)-trajectory \( t \to (x^*(t), t/2) \) takes values in \( A \) and achieves zero cost. It follows that

\[ V(0; (0,0)) = 0. \quad (12) \]
For each $k \geq 1$ let $s_k$ be the point in $[t_{k+1}, t_k]$ at which $x^*(\cdot)$ takes value zero. A direct computation yields

$$s_k = \frac{t_k}{2} = \frac{1}{2} \frac{1}{3^k}.$$ 

Hence there exists a constant $c$ such that

$$k \geq c \ln s_k$$

for all $k \geq 1$. But for any feasible $F$-trajectory $(x(\cdot), y(\cdot))$ on $[s_k, 1]$ such that $(x(s_k), y(s_k)) = (0, 0)$ we have $|x(t)| \leq y(t) \leq \frac{t - s_k}{2}$. Therefore

$$V(s_k; (0, 0)) \geq \inf \left\{ \| \dot{x} - \dot{x}^* \|_{L^1(s_k, 1)} : x(s_k) = 0, |x(t)| \leq \frac{t - s_k}{2} \text{ for all } t \in [s_k, 1] \right\} \quad (13)$$

for all $k \geq 1$. For any arc $x(\cdot)$ on $[s_k, 1]$, such that $x(s_k) = 0$ and $|x(t)| \leq \frac{t - s_k}{2}$, we have

$$\| \dot{x}(\cdot) - \dot{x}^*(\cdot) \|_{L^1(s_k, 1)} = \int_{s_k}^{t_k} |\dot{x}(t) - \dot{x}^*(t)| \, dt + \sum_{i=1}^{k} \int_{t_i}^{t_{i-1}} |\dot{x}(t) - \dot{x}^*(t)| \, dt \geq \sum_{i=1}^{k} |(x^*(t_{i-1}) - x(t_{i-1})) - (x^*(t_i) - x(t_i))| \geq k \cdot s_k \geq c s_k \cdot |\ln(s_k)|.$$ 

It follows from the definition of the $s_k$’s, (12) and (13) that $z_k = (s_k; (0, 0)) \to (0, 0, 0)$ and

$$|V(z_k) - V(0)| \geq c |z_k| \cdot |\ln |z_k||.$$ 

Consequently the structure of the regularity estimate in Proposition 3.1-(i) is optimal, in the sense that, for any $K' > 0$, $\epsilon > 0$ and modulus $\theta(\cdot)$ satisfying

$$\lim_{h \downarrow 0} \theta(h)/(h |\ln h|) = 0,$$ 

there exists $z \in A$ such that $|z| \leq \epsilon$ and $V(z) - V(0) > K'\theta(|z|)$.

### 4 Proof of Theorem 2.1

We shall first prove a restricted version of Thm. 2.1 in which additional hypotheses are imposed and in which the feasible arc $x(\cdot)$ approximating the reference $F$-trajectory is required to be merely a $coF$-trajectory. The final stage of the proof will be to remove these restrictions. Define

$$e_1 = (1, 0, \ldots, 0) \quad \text{and} \quad e_2 = (0, 1, 0, \ldots, 0).$$

**Proposition 4.1** Assume that the compact set $F \subset \mathbb{R}^n$ and the cone $A = \{x \in \mathbb{R}^n | b_1 \cdot x \leq 0, b_2 \cdot x \leq 0 \}$ satisfy, in addition to (H1), the hypotheses

(S1): $-b_1 = e_1$ and $-b_2 = e_2,$

(S2): there exist numbers $a_1 > 0$ and $a_2 > 0$, and a vector $\bar{v} \in coF$ such that $a_1^2 + a_2^2 = 1$, $\bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ with $\bar{v}_1 > 0$ and $\bar{v}_2 > 0$ and, defining $a = (a_1, a_2, 0, \ldots, 0),$ we have

$$a \cdot \bar{v} = \max_{w \in coF} a \cdot v.$$
Then there exist constants $K > 0$ with the following property: take any $t_0 \in [0, 1]$ and any $coF$-trajectory $\hat{x}(\cdot)$ on $[t_0, 1]$ satisfying $h(\hat{x}) > 0$. Assume that 

$$(T1): \hat{x}(t_0) \in A,$$

$$(T2): a \cdot \hat{x}(t) > 0 \quad \text{for all } t \in [t_0, 1].$$

Then there exists a $coF$-trajectory $x(\cdot)$ such that $x(t_0) = \hat{x}(t_0)$, $x(t) \in A$ for all $t \in [t_0, 1]$,

$$\dot{x}(t) = \hat{v} \quad \text{for a.e. } t \in \{s \in [t_0, 1] : \hat{x}(s) \neq \hat{x}(s)\},$$

$$d_{[t_0,1]}(\hat{x}, \dot{x}) \leq K (1 + |\ln(a \cdot \hat{x}(t_0))|) h(\hat{x}).$$

If, in addition to the other hypotheses, the strict convexity hypothesis $(H2)$ of Thm. 2.1 is in force, then (for some possibly larger value of the constant $K$)

$$d_{[t_0,1]}(\hat{x}, \dot{x}) \leq K h(\hat{x}).$$

The proof of Proposition 4.1 will be based on certain implications of the hypotheses $(H1)$ and (S1), and on properties of trajectories confined to $A$. These are summarized in the following lemma, whose proof is straightforward. Take $a = (a_1, a_2, 0, \ldots, 0)$ as in (S2) of Proposition 4.1 and consider the sets

$$H \doteq \{x \in \mathbb{R}^n : a \cdot x = 0\}, \quad A^+(x) \doteq (x + H) \cap A.$$

**Lemma 4.2** Take a compact set $F$ and a cone $A$ having the representation $(4)$. Let the assumptions $(H1)$, (S1), and (S2) hold, and let $a$ be the vector in (S2).

(i): Let $w, x : [t_1, t_2] \to \mathbb{R}^n$ be continuous functions such that $w(t), x(t) \in A$ for all $t \in [t_1, t_2]$. Assume that

either “$b_1 \cdot x(t_1) = 0$ and $b_2 \cdot x(t_2) = 0$” or “$b_2 \cdot x(t_1) = 0$ and $b_1 \cdot x(t_2) = 0$,”

and moreover $a \cdot x(t) \geq a \cdot w(t)$ for all $t \in [t_1, t_2]$.

Then there exists $t' \in [t_1, t_2]$ and $\eta \in \{(0, 0)\} \times \mathbb{R}^{n-2}$ such that

$$x(t') = w(t') + [a \cdot (x(t') - w(t'))] a + \eta.$$

(ii): There exists $c \in (0, 1)$ such that, for any $coF$-trajectory $x(\cdot)$ on $[t_1, t_2]$ with $x(t) \in A$ for all $t \in [t_1, t_2]$ and any $\bar{a} \geq 0$ satisfying $a \cdot x(t_1) \geq \bar{a}$, the assumption

“$b_1 \cdot x(t_1) \leq b_2 \cdot x(t_1)$ and $b_1 \cdot x(t_2) = 0$” or “$b_2 \cdot x(t_1) \leq b_1 \cdot x(t_1)$ and $b_2 \cdot x(t_2) = 0$”

implies $t_2 - t_1 \geq c \bar{a}$.

(iii): There exist constants $c_1 > 0$ and $c_2 > 0$ with the following property. Take any $t_0 \in [0, 1]$, any $coF$-trajectory $x^*(\cdot)$ on $[t_0, 1]$ such that $a \cdot x^*(t_0) > 0$, any number $\xi \geq 0$, and any vector $\eta \in \{(0, 0)\} \times \mathbb{R}^{n-2}$. Define

$$z(t) = x^*(t) + \left(\pi_{A^+(x^*(t_0))}(x^*(t_0)) - x^*(t_0)\right) + \xi a + \eta.$$

Then

$$h(z) \leq \left((1 + c_1)h(x^*) - c_2 \xi\right) \vee 0.$$

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Next we provide a simple, self-contained proof of the special case of the 'one-constraint distance estimate' of [3, Thm. 2.1], where the velocity set $F$ does not depend on $(t,x)$.

**Lemma 4.3 (Distance Estimates for a Single State Constraint)** Take any $\bar{F} \subset \mathbb{R}^n$, and any vector $b \in \mathbb{R}^n$ with $|b| = 1$. Assume that there exists $\tilde{\epsilon} > 0$ and $\bar{v} \in \bar{F}$ such that

$$b \cdot \bar{v} = -\tilde{\epsilon}.$$ 

Take any $t_0 \in [0,1]$ and $\bar{F}$-trajectory $\dot{x}$ on $[t_0,1]$ such that $b \cdot \dot{x}(t_0) \leq 0$. Then there exists an $\bar{F}$-trajectory $x(\cdot)$ on $[t_0,1]$ such that $x(t_0) = \bar{x}(t_0)$, $b \cdot x(t) \leq 0$ for all $t \in [t_0,1]$,

$$\{t \in [t_0,1] | \dot{x}(t) \neq \dot{\bar{x}}(t) \} \subset \{t \in [t_0,1] | b \cdot \dot{x}(t) > 0\}$$ 

(modulo a null-set) and

$$d_{[t_0,1]}(\dot{x}, \dot{\bar{x}}) \leq \bar{K} h^+$$

where $\bar{K} = (1/\tilde{\epsilon})$ and $h^+ = \max_{t \in [t_0,1]} \{b \cdot x(t) \lor 0\}$.

**Proof.** We can assume that $b \cdot \dot{x}(t) > 0$ for some $t$. We then define

$$t' = \inf \{t \in [t_0,1] | b \cdot \dot{x}(t) > 0\}, \quad \mathcal{A} = \{t \in [t',1] | b \cdot \dot{x} > 0\}.$$ 

If $\text{meas}(\mathcal{A} \cap [t',1]) \geq (1/\tilde{\epsilon})h^+$, we also define

$$\bar{\epsilon} = \inf \{t \in [t',1] | \text{meas}(\mathcal{A} \cap [t', t]) = (1/\tilde{\epsilon})h^+\};$$

otherwise we set $\bar{\epsilon} = 1$.

Now choose $x(\cdot)$ to be the $\bar{F}$-trajectory on $[t_0,1]$ satisfying $x(t_0) = \bar{x}(t_0)$ and

$$\begin{align*}
\dot{x}(t) = \begin{cases}
\bar{v} & \text{if } t \in [t_0,\bar{\epsilon}] \cap \mathcal{A}, \\
\dot{\bar{x}}(t) & \text{otherwise}.
\end{cases}
\end{align*}$$

It is now straightforward to check that the trajectory $x(\cdot)$ constructed in this way satisfies all the asserted properties.

**Proof of Prop. 4.1.** Take $a$ and $\bar{v}$ as in (S2). $\kappa$ is the bound of (7). Define the positive numbers $\alpha \doteq a \cdot \bar{v} > 0$ and $\bar{\epsilon} \doteq \min\{\bar{v}_1, \bar{v}_2\}$.

Choose any $\alpha' \in (0, \alpha)$. If (H2) is satisfied, we choose $\alpha'$ additionally to satisfy

$$a \cdot \bar{v} > \alpha' > \max_{v \in \co F} \left\{ a \cdot v \mid \text{either } b_1 \cdot v \geq 0 \text{ or } b_2 \cdot v \geq 0 \right\}.$$ 

Set $\xi_0 = 0$ and $\eta_0 = (0, \ldots, 0)$. We shall now describe the construction of numbers $t_1 > t_0$, $t_0 \geq t_0$, $\xi_1 \geq \xi_0$, a vector $\eta_1 = (0,0, \eta^{(3)}_1, \ldots, \eta^{(n)}_1)$ and a coF-trajectory $x_0 : [t_0,1] \to \mathbb{R}^n$ such that $x_0(t_0) = \pi_{A^+(\dot{x}(t_0))}(\dot{x}(t_0)) + \xi_0 a + \eta_0$.

We can assume, without loss of generality, that $\dot{x}(t_0)$ is closer to the 'upper' boundary of $A$ that the 'lower' boundary, i.e. $b_1 \cdot \dot{x}(t_0) \geq b_2 \cdot \dot{x}(t_0)$.

Define the co $F$-trajectory $z$ on $[t_0,1]$ to be

$$z(t) = \dot{x}(t) + \left( \pi_{A^+(\dot{x}(t_0))}(\dot{x}(t_0)) - \bar{x}(t_0) \right) + \xi_0 a + \eta_0.$$ 

(17)
By Lemma 4.2 (iii) we can find positive constants $c_1$ and $c_2$ such that
\[
h(z) \leq \left( (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right) \vee 0 = (1 + c_1)h(\dot{x}) - c_2 \xi_0.
\] (18)

(Notice that $(1 + c_1)h(\dot{x}) - c_2 \xi_0 = (1 + c_1)h(\dot{x}) > 0$). Set $c^* := 1 \vee \frac{1}{c_2(\alpha - \alpha')}$. Define
\[
A_1 := \{ t \in [t_0, 1] \mid a \cdot \dot{x}(t) \leq \alpha' \},
\]
\[
t_0 := (\inf \{ t \mid \text{meas} ([t_0, t] \cap A_1) = c^* [(1 + c_1)h(\dot{x}) - c_2 \xi_0] \}) \wedge 1.
\]
(The infimum is taken to be $+\infty$ if the condition on $t$ cannot be satisfied.)

Let $y_1$ be the co$F$-trajectory on $[t_0, 1]$ satisfying $y_1(t_0) = z(t_0)$ and
\[
\dot{y}_1(t) = \begin{cases} 
\bar{v} & \text{if } t \in A_1 \cap [t_0, t_0] \\
\dot{z}(t) & \text{otherwise}.
\end{cases}
\]

Noting that $\dot{z}(t) = \dot{x}(t)$ a.e. $t \in [t_0, 1]$, we see that
\[
d_{[t_0, 1]}(\dot{y}_1, \dot{x}) \leq c^* \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right],
\]
\[
h(y_1) \leq h(z) + ||z - y_1||_{L_\infty} \leq c^* (1 + 2\kappa) \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right].
\]

Now apply Lemma 4.3 with $b = b_1$, $\bar{F} = co F$ (notice that $\bar{v} \in \bar{F}$) and with reference trajectory $y_1$ to obtain a co$F$-trajectory $y_2$ on $[t_0, 1]$ such that $y_2(t_0) = y(t_0),
\]
\[
b_1 \cdot y_2(t) \leq 0 \quad \text{for all } t \in [t_0, 1], \quad \text{and}
\]
\[
d_{[t_0, 1]}(\dot{y}_1, \dot{y}_2) \leq c^* \bar{e}^{-1}(1 + 2\kappa) \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right].
\]

We have
\[
d_{[t_0, 1]}(\dot{y}_2, \dot{x}) \leq c^* (1 + \bar{e}^{-1}(1 + 2\kappa)) \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right] \text{ and}
\]
\[
h(y_2) \leq h(y_1) + ||y_1 - y_2||_{L_\infty} \leq c^* (1 + 2\kappa \bar{e}^{-1})(1 + 2\kappa) \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right].
\]

If $b_2 \cdot y_2(t) \leq 0$ for all $t \in [t_0, 1]$, then we take $t_1 = 1$, $\xi_1 = \xi_0$, $\eta_1 = \eta_0$ and $x_0(.) = y_2(.)$.

Otherwise we can define $s_1 \in [t_0, 1]$ to be
\[
s_1 = \inf \{ t \in [t_0, 1] \mid b_2 \cdot y_2(t) > 0 \}.
\]

Clearly $b_2 \cdot y_2(s_1) = 0$. Apply Lemma 4.3 again, with $b = b_2$, $\bar{F} = co F$ (recall that $\bar{v} \in \bar{F}$) and reference trajectory $y_2$ restricted to $[s_1, 1]$. This yields a co$F$-trajectory $y_3$ on $[t_0, 1]$ such that $y_3(t) = y_2(t)$ for $t \in [t_0, s_1]$, $b_2 \cdot y_3(t) \leq 0$ for all $t \in [s_1, 1]$ and
\[
d_{[t_0, 1]}(\dot{y}_2, \dot{y}_3) \leq c^* \bar{e}^{-1}(1 + 2\kappa \bar{e}^{-1})(1 + 2\kappa) \left[ (1 + c_1)h(\dot{x}) - c_2 \xi_0 \right].
\]

If $b_1 \cdot y_3(t) \leq 0$ for all $t \in [s_1, 1]$, then we take $t_1 = 1$, $\xi_1 = \xi_0$, $\eta_1 = \eta_0$ and $x_0(.) = y_2(.)$.

Otherwise, there exists $s_2 \in [s_1, 1]$ such that $y_3(t) \in A$ for all $t \in [s_1, s_2]$ and $b_1 \cdot y_3(s_2) = 0$. Set $B_1 := \{ s \in [t_0, s_2] \mid \dot{y}_3(s) \neq \dot{x}(s) \}$. Notice that we have $\dot{y}_3(s) = \bar{v}$ for a.e. $s \in B_1$ and, therefore, from hypothesis (S2)
\[
a \cdot (\dot{y}_3(s) - \dot{x}(s)) \geq 0 \quad \text{for a.e. } s \in B_1.
\]
We now apply Lemma 4.2(i) on \([s_1,s_2]\), identifying
\[
w(t) = \pi_{A^+([\hat{x}(t)])}(\hat{x}(t)) + \xi_0 a + \eta_0 \quad \text{and} \quad x(t) = y_3(t) .
\]
This is possible, because \(b_2 \cdot y_3(s_1) = 0\) and \(b_1 \cdot y_3(s_2) = 0\), \(a \cdot [z(t) - w(t)] = 0\) on \([s_1,s_2]\) and since (in view of the facts that \(y_3(t_0) = w(t_0)\) and \(\frac{d}{dt} a \cdot w(t) = \frac{d}{dt} a \cdot \hat{x}(t)\) a.e.)
\[
a \cdot [y_3(t) - w(t)] = a \cdot [y_3(t) - z(t)] = \int_{[t_0,t]} a \cdot (\dot{y}_3(s) - \dot{x}(s)) ds = \int_{[t_0,t] \cap B_1} a \cdot (\dot{v} - \dot{x}(s)) ds \geq 0
\]
for all \(t \in [s_1,s_2]\). The Lemma yields points \(t_1 \in [s_1,s_2]\) and \(\eta_1 \in \{(0,0) \times \mathbb{R}^{n-2}\}\) such that
\[
y_3(t_1) = \pi_{A^+([\hat{x}(t_1)])}(\hat{x}(t_1)) + \xi_1 a + \eta_1 ,
\]
where
\[
\xi_1 = \xi_0 + \int_{[t_0,t_1]} a \cdot (\dot{y}_3(t) - \dot{x}(t)) dt .
\]
Now define the \(co F\)-trajectory \(x_0\) on \([t_0,1]\):
\[
x_0(t) = y_3(t), \quad \text{for all } t \in [t_0,t_1] \quad \text{and} \quad x_0(t) = \hat{x}(t) + \left( \pi_{A^+([\hat{x}(t_1)])}(\hat{x}(t_1)) - \hat{x}(t_1) \right) + \xi_1 a + \eta_1 , \quad \text{for all } t \in [t_1,1] .
\]
We have
\[
d_{[t_0,t_1]}(\hat{x}_0, \hat{x}) \leq k_1 \left[(1 + c_1)h(\hat{x}) - c_2 \xi_0 \right] ,
\]
where \(k_1 \doteq c^* (1 + 2\kappa)(1 + \varepsilon^{-1})(1 + 2\kappa \varepsilon^{-1})\). From (19) and Lemma 4.2 it follows that
\[
h(x_0) \leq \left((1 + c_1)h(\hat{x}) - c_2 \xi_1 \right) \vee 0 .
\]
If \((1 + c_1)h(\hat{x}) - c_2 \xi_1 \leq 0\), then \(x_0(t) \in A\) for all \(t \in [t_0,1]\) and we redefine \(t_1 = 1\). (Notice that (20) continues to be satisfied for the re-defined \(t_1\)).

Provided \(t_1 < 1\), we can repeat the above construction, with \((t_0, \xi_0, \eta_0)\) replacing \((t_1, \xi_1, \eta_1)\), to obtain elements \(t_2, \bar{t}_1, \xi_2, \eta_2\) and \(x_1(.) : [t_1,1] \to \mathbb{R}^n\). There results a sequence
\[
\left\{ \left(t_{i+1}, \bar{t}_1, \xi_{i+1}, \eta_{i+1} \in \{(0,0) \times \mathbb{R}^{n-2}\}, \ x_1(.) : [t_i,1] \to \mathbb{R}^n \right) \right\}^{N-1}_{i=0} .
\]
The recursion ends with the construction of \(x_{N-1}(.)\) and \(t_N\), where \(N\) is the first value of the index \(i\) such that \(t_i = 1\).

Notice that the construction is such that \(a \cdot \hat{x}_i(t) \geq a \cdot \hat{x}(t)\) a.e. Since \(a \cdot x_i(t_i) \geq a \cdot \hat{x}(t_i)\) and, by hypothesis (T2), \(a \cdot x_i(t)\) is uniformly bounded away from zero, for all \(i = 0, \ldots, N - 1\). It follows from Lemma 4.2(ii) that the lengths of the intervals \([t_i, t_{i+1}]\) admit a uniform positive lower bound. Satisfaction of the stopping condition \(t_N = 1\) after a finite number of steps is therefore guaranteed.
Summarizing we have \( x_0(t_0) = \dot{x}(t_0), x_1(t_1) = x_0(t_1), \ldots, x_{N-1}(t_{N-1}) = x_{N-2}(t_{N-1}) \) and, for \( i = 0, \ldots, N - 1, \)

\[
x_i(t_i) = \pi_{A_i}(\hat{x}(t_i))(\hat{x}(t_i)) + \xi_i + \eta_i, \tag{21}
\]

\[
x_i(t) = \hat{x}(t) + \left( \pi_{A_i}(\hat{x}(t_i))(\hat{x}(t_i)) - \hat{x}(t_i) \right) + \xi_i + \eta_i + 1, \quad \text{for all } t \in [t_i, 1], \tag{22}
\]

\[
\xi_{i+1} = \xi_i + \int_{[t_i, t_i+1]} a \cdot (\hat{x}(t) - \hat{x}(t))dt, \tag{23}
\]

\[
(1 + c_1)h(\hat{x}) - c_2\xi_i > 0, \quad t_i < 1, \quad x_i(t) \in A \quad \text{for all } t \in [t_i, t_i+1],
\]

\[
\bar{\bar{t}}_i = \left( \inf \{ t \mid \text{meas}(\{ t_i, t \} \cap A_i) = c^* \left[ (1 + c_1)h(\hat{x}) - c_2\xi_i \right] \} \right) \cup 1,
\]

\[
d_{[t_i, t_i+1]}(\hat{x}_i, \hat{\hat{x}}) \leq k_1 [(1 + c_1)h(\hat{x}) - c_2\xi_i], \tag{24}
\]

\[
a \cdot \hat{x}_i(t) \geq a' \quad \text{for a. e. } t \in [t_i, \bar{\bar{t}}_i]. \tag{25}
\]

Define \( x(.) \) to be the concatenation of \( x_i(.) \) restricted to \([t_i, t_i+1], i = 0, \ldots, N - 1:\)

\[
x(t) = \sum_{i=0}^{N-2} \chi_{[t_i, t_i+1]}(t)x_i(t) + \chi_{[t_{N-1}, t_N]}x_{N-1}(t), \tag{26}
\]

(where \( \chi_I \) denotes the indicator of the interval \( I \)). \( x(.) \) constructed in this way is a co \( F \)-trajectory satisfying \( x(t_0) = \hat{x}(t_0) \) and \( x(t) \in A \) for all \( t \in [t_0, 1] \).

Now suppose that the ‘strict convexity’ hypothesis \((H2)\) is in force. We shall verify the linear estimate \((16)\). We deduce from this extra hypothesis and \((S2)\) that \( a \cdot (\hat{x}(t) - \hat{x}(t)) \geq a' \) a.e. on \( \{ t' \in [t_0, 1] \mid \hat{x}(t') \neq \hat{x}(t') \} \). From \((23)\) it then follows that \n
\[
\xi_{i+1} - \xi_i \geq a'd_{[t_i, t_i+1]}(\hat{x}_i, \hat{\hat{x}})
\]

for \( i = 0, 1, \ldots, N - 2 \). Iterating this inequality and using \((24)\), one obtains

\[
d_{[t_{N-1}, t_N]}(\hat{x}, \hat{\hat{x}}) \leq k_1(1 + c_1)h(\hat{x}) - k_1c_3a'd_{[t_0, t_{N-1}]}(\hat{x}, \hat{\hat{x}}).
\]

We have established the linear estimate \((16)\)

\[
d_{[t_0, 1]}(\hat{x}, \hat{\hat{x}}) \leq Kh(\hat{x})
\]

with \( K = k_1(1 + c_1)/(1 \land k_1c_3a') \).

Suppose henceforth, then, that \((H2)\) is not satisfied. Notice that if for some \( i \in \{0, 1, \ldots, N - 1\} \) we have \( t_i \leq t_{i+1}, \) then \( i + 1 = N \). This follows from the estimates

\[
\xi_{i+1} - \xi_i \geq \int_{[t_i, t_{i+1}]} a \cdot (\hat{x}(t) - \hat{x}(t))dt \\
\geq (\alpha - a') \text{meas} \left( \{ t \in [t_i, t_{i+1}] \mid a \cdot \hat{x}(t) \leq a' \} \right) \\
= (\alpha - a')c^* \left[ (1 + c_1)h(\hat{x}) - c_2\xi_i \right] \\
\geq \frac{1}{c_2} \left[ (1 + c_1)h(\hat{x}) - c_2\xi_i \right],
\]

which implies \( c_2\xi_{i+1} \geq (1 + c_1)h(\hat{x}) \) and so, in view of \((22)\) and Lemma 4.2 (iii),

\[
h(x_i) \leq (1 + c_1)h(\hat{x}) - c_2\xi_{i+1} \leq 0.
\]
But this means \( x_i(t) \in A \) for all \( t \in [t_i, 1] \), whence \( t_{i+1} = 1 \).

Hence, we continue assuming \( t_i > t_{i+1} \) for all \( i = 0, \ldots, N-2 \) (we can also assume that \( N \geq 3 \)). Recalling the definition of the \( t_i \)'s and from (25), we conclude that, for each \( i = 0, \ldots, N-2 \),

\[
a \cdot \dot{x}_i(t) \geq \alpha' \quad \text{for all} \quad t \in [t_i, t_{i+1}].
\]

It follows that

\[
a \cdot x(t_{i+1}) - a \cdot x(t_i) \geq \alpha'(t_{i+1} - t_i).\]

But, since \( a \cdot x(t_i) \geq a \cdot \dot{x}(t_i) \geq \dot{a} \) (for some \( \dot{a} > 0 \)), from (T2) and Lemma 4.2 (ii) we know

\[
t_{i+1} - t_i \geq c a \cdot x(t_i).
\]

Eliminating \( t_{i+1} - t_i \) from these relations yields

\[
a \cdot x(t_{i+1}) \geq (1 + c \alpha') a \cdot x(t_i).
\]

But then, since \( x(t_0) = \dot{x}(t_0) \), we deduce that \( a \cdot x(t_i) \geq (1 + c \alpha')^i a \cdot \dot{x}(t_0) \), whence

\[
(t_{i+1} - t_i) \geq c(1 + c \alpha')^i a \cdot \dot{x}(t_0).
\]

Therefore,

\[
1 \geq t_{N-1} - t_0 \geq \sum_{i=0}^{N-2} (t_{i+1} - t_i) = (\alpha')^{-1} ((1 + c \alpha')^{N-1} - 1) \cdot a \cdot \dot{x}(t_0)
\]

and, consequently,

\[
(1 + c \alpha')^{N-1} \leq 1 + \alpha'(a \cdot \dot{x}(t_0))^{-1}.
\]

This inequality implies the existence of a constant \( K' \) (whose value depends only on \( c \) and \( \alpha' \)) such that

\[
N \leq K'(1 + |\ln(a \cdot \dot{x}(t_0))|).
\]

We deduce now from (24) that

\[
d_{[t_0,1]}(\dot{x}, \ddot{x}) \leq k_1(1 + c_1)h(\dot{x}) \times N \leq K(1 + |\ln(a \cdot \dot{x}(t_0))|)
\]

where \( K = k_1(1 + c_1)K' \).

Finally, notice that from the construction of \( x(\cdot) \) clearly also (14) follows. This concludes the proof.

The next lemma asserts the existence of a co\( F \)-trajectory on \([t_0, 1] \), \( x^*(\cdot) \), close to the nominal \( F \)-trajectory \( \dot{x}(\cdot) \), whose distance from the edge is, eventually, bounded below by the constraint violation \( h(\ddot{x}) \).

**Lemma 4.4** Assume (H1), (S1) and (S2). Take any \( t_0 \in [0,1] \) and an \( F \)-trajectory \( \dot{x} \) on \([t_0,1] \) satisfying (T1) and with \( h(\dot{x}) > 0 \). Then, there exist a constant \( K^* > 0 \) (that depends only on the problem data) and a co\( F \)-trajectory \( x^*(\cdot) \) on \([t_0,1] \) such that \( x^*(t_0) = \dot{x}(t_0) \) and

\[
x^*(t) \in A \quad \text{for all} \quad t \in [t_0, (t_0 + \alpha^{-1}h(\dot{x})) \land 1], \quad \text{(27)}
\]

\[
a \cdot x^*(t) \geq h(\dot{x}) \quad \text{for all} \quad t \in ((t_0 + \alpha^{-1}h(\dot{x})) \land 1, 1], \quad \text{(28)}
\]

\[
d_{[t_0,1]}(\dot{x}, \ddot{x}) \leq K^*h(\dot{x}), \quad \text{(29)}
\]

\[
\dot{x}^*(t) = \ddot{v} \quad \text{for a.e.} \ t \in \{s \in [t_0, 1] : \dot{x}^*(s) \neq \dot{x}(s)\}. \quad \text{(30)}
\]

Here, \( \alpha = a \cdot \ddot{v} \) where the vectors \( a \) and \( \ddot{v} \) are as in (S2).
Proof. We assume (H1), (S1) and (S2). Take any }t_0\in [0,1]\text{ and } F\text{-trajectory } \dot{x}(t) \text{ on } [t_0,1]\text{ satisfying (T1) but not necessarily (T2). Let } y(\cdot) \text{ be the co } F\text{-trajectory satisfying } y(t_0) = \dot{x}(t_0) \text{ and }$

\dot{y}(t) = \begin{cases}
\ddot{v} & \text{if } t \in [t_0, (t_0 + \alpha^{-1}h(\dot{x})) \wedge 1] \\
\dot{x}(t) & \text{otherwise}.
\end{cases}$

Note that, since } y(t_0) \in A \text{ and } b_j \cdot \ddot{v} \leq -\min\{\ddot{v}_1, \ddot{v}_2\} < 0 \text{ for } j = 1, 2, \text{ we have that } y(t) \in A \text{ for all } t \in [t_0, (t_0 + \alpha^{-1}h(\dot{x})) \wedge 1]. \text{ If } t_0 + \alpha^{-1}h(\dot{x}) \geq 1, \text{ then (27)-(30) are satisfied with } K^* = \alpha^{-1} \text{ for the choice } x^*(\cdot) = y(\cdot), \text{ since } y(\cdot) \text{ satisfies the state constraint and } d_{[t_0,1]}(\dot{x}, \dot{y}) \leq \alpha^{-1}h(\dot{x}). \text{ We may assume therefore that } t_0 + \alpha^{-1}h(\dot{x}) < 1. 

For } \tau \in [t_0 + \alpha^{-1}h(\dot{x}), 1] \text{ define the co } F\text{-trajectory } x(\cdot; \tau) \text{ on } [t_0, 1] \text{ to satisfy } x(t; \tau) = y(t) \text{ on } [t_0, t_0 + \alpha^{-1}h(\dot{x})] \text{ and }$

\frac{d}{dt} x(t; \tau) = \begin{cases}
\ddot{v} & \text{if } t \in B \cap [t_0 + \alpha^{-1}h(\dot{x}), \tau], \\
\dot{x}(t) & \text{otherwise},
\end{cases}$

where$
B = \left\{ t \in [t_0 + \alpha^{-1}h(\dot{x}), 1] \mid a \cdot \dot{x}(t) \leq \frac{1}{2} \alpha \right\}.$

Since } a \cdot y(t_0) \geq 0 \text{ (this follows from the fact that } y(t_0) \in A), \text{ and since } \frac{d}{dt} a \cdot x(\cdot; \tau) \text{ is non-negative on } [t_0, \tau] \text{ and exceeds } \alpha \text{ on } [t_0, t_0 + \alpha^{-1}h(\dot{x})], \text{ we have }$

a \cdot x(t; \tau) \geq h(\dot{x}) \text{ for all } t \in [t_0 + \alpha^{-1}h(\dot{x}), \tau]. \tag{31}$

We can thus define$
\tau^* = \min \left\{ \tau \in [t_0 + \alpha^{-1}h(\dot{x}), 1] \mid a \cdot x(t; \tau) \geq h(\dot{x}) \text{ for all } t \in [t_0 + \alpha^{-1}h(\dot{x}), 1] \right\}. \tag{32}$

Since } a \cdot x(t; 1) > h(\dot{x}) \text{ for all } t \in (t_0 + \alpha^{-1}h(\dot{x}), 1], \text{ we can assume that } \tau^* < 1. \text{ We can also suppose that } \tau^* > t_0 + \alpha^{-1}h(\dot{x}), \text{ otherwise } x^*(\cdot) = y(\cdot) \text{ already satisfies the required relations (27)-(30). Now set } x^*(\cdot) = x(\cdot; \tau^*). \text{ We have } x^*(t) \in A \text{ for } t \in [t_0, t_0 + \alpha^{-1}h(\dot{x})], \text{ while }$

a \cdot x^*(t) \geq h(\dot{x}) \text{ for } t \in [t_0 + \alpha^{-1}h(\dot{x}), 1].$

If we can find a constant } K' > 0 \text{ such that }$
\text{meas } (B \cup [t_0 + \alpha^{-1}h(\dot{x}), \tau^*]) \leq K'h(\dot{x}), \tag{33}$

then (28)-(30) are confirmed with } K^* = K' + \alpha^{-1}, \text{ since }$
d_{[t_0,1]}(\dot{x}, \dot{x}^*) \leq \alpha^{-1}h(\dot{x}) + \text{meas } (B \cap [t_0 + \alpha^{-1}h(\dot{x}), \tau^*]). \tag{34}$

Note that, since } y \text{ and } \dot{x} \text{ differ on a set of measure at most } \alpha^{-1}h(\dot{x}) \text{ and } b_j \cdot \dot{x}(t) \leq h(\dot{x}) \text{ on } [t_0, 1], j = 1, 2, \text{ we have }$
a \cdot y(t) \geq -\tilde{K}h(\dot{x}) \text{ for } t \in [t_0, 1],$

where } \tilde{K} = a_1 + a_2 + 2\alpha^{-1}\kappa. \text{ For all } t \in [\tau^*, 1]$

\begin{align*}
a \cdot (x^*(t) - y(t)) &= \int_{[t_0 + \alpha^{-1}h(\dot{x}), t]} a \cdot (\dot{x}^*(s) - \dot{y}(s)) \, ds \\
&= \int_{B \cap [t_0 + \alpha^{-1}h(\dot{x}), \tau^*]} a \cdot (\ddot{v} - \dot{y}(s)) \, ds \\
&\geq \frac{\alpha}{2} \text{meas } (B \cap [t_0 + \alpha^{-1}h(\dot{x}), \tau^*]), \tag{35}
\end{align*}
From (30) and (38) we also have

\[ a \cdot x^*(t) \geq - \tilde{K} h(\hat{x}) + \frac{\alpha}{2} \text{meas}\left( B \cap [t_0 + \alpha^{-1}h(\hat{x}), \tau^*] \right). \]

It follows from this relation and (31) (with \( \tau = \tau^* \)) that

\[ h(\hat{x}) \geq - \tilde{K} h(\hat{x}) + (\alpha/2) \text{meas}\left( B \cap [t_0 + \alpha^{-1}h(\hat{x}), \tau^*] \right), \]

for otherwise the value of \( \tau^* \) could be reduced, in contradiction with the definition (32). This implies (33) with \( K' = (2/\alpha)(1 + \tilde{K}) \). And the statement of the lemma is proved.

**Proof of Theorem 2.1.**

We first confirm the assertions of the theorem under the additional hypothesis (S1) and when the initial time \( t_0 \in [0, 1] \) and nominal \( F \)-trajectory \( \hat{x} \) on \([t_0, 1]\) satisfies the additional condition \( \hat{x}(t_0) \in A \) (that is (T1)). We can assume that \( h(\hat{x}) > 0 \) and \( t_0 < 1 \), since otherwise the assertions are trivial.

Notice that hypotheses (S1) and (H1) imply (S2). This fact is proved in the appendix (see Lemma 4.5).

Then we can apply Lemma 4.4 to the reference \( F \)-trajectory \( \hat{x}(\cdot) \) on \([t_0, 1]\), obtaining a constant \( K^* > 0 \) (which depends only on the data of the problem) and a \( \text{co}F \)-trajectory \( x^*(\cdot) \) such that (27) – (30) are satisfied. Write

\[ t_0^* = t_0 + \alpha^{-1}h(\hat{x}). \]

If \( t_0^* \geq 1 \), then the linear estimate (16) is satisfied with \( K = K^* \) for the choice \( x(\cdot) = x^*(\cdot) \), since \( x^*(\cdot) \) satisfies the constraint on \([t_0, 1]\) and \( d_{[t_0, 1]}(\hat{x}, \hat{x}^*) \leq K^* h(\hat{x}) \). Thus we can assume that \( t_0^* < 1 \). Now we observe that, owing to Lemma 4.4 (see (28)), the \( \text{co}F \)-trajectory \( x^*(\cdot) \) restricted to \([t_0^*, 1]\) satisfies both (T1) and (T2) of Prop. 4.1 (the new initial condition is \( x_0^* = x^*(t_0^*) \in A \)).

Notice also that from (29) we have

\[ h(x^*) \leq (1 + 2\kappa K^*) h(\hat{x}). \quad (36) \]

(Here \( \kappa \) is the constant in (7).)

Consequently, applying Prop. 4.1 to the reference \( \text{co}F \)-trajectory \( x^*(\cdot) \) on \([t_0^*, 1]\), we can find a constant \( K_1 > 0 \) (which depends only on the problem data) and a \( \text{co}F \)-trajectory \( x_1(\cdot) \) on \([t_0^*, 1]\) such that

\[ d_{[t_0^*, 1]}(\hat{x}_1, \hat{x}^*) \leq K_1 \left( 1 + |\ln(a \cdot x^*(t_0^*))| \right) h(x^*), \quad (37) \]

\[ \hat{x}_1(t) = \vec{v} \quad \text{for a.e. } t \in \{ s \in [t_0^*, 1] : \hat{x}_1(s) \neq \hat{x}^*(s) \}. \quad (38) \]

Observe that if \( a \cdot x^*(t_0^*) \geq 1 \) then, (36) and (37) immediately yield a linear estimate (as in (16)). So we may assume that \( a \cdot x^*(t_0^*) < 1 \) and, from (28), (36) and (37), we have

\[ d_{[t_0^*, 1]}(\hat{x}_1, \hat{x}^*) \leq K_1 \left( 1 + 2\kappa K^* \right) \left( 1 + |\ln(h(\hat{x}))| \right) h(\hat{x}). \quad (39) \]

Concatenating \( x^*(\cdot) \) restricted to \([t_0, t_0^*]\) with \( x_1(\cdot) \) on \([t_0^*, 1]\), we obtain a \( \text{co}F \)-trajectory \( x_2(\cdot) \) on \([t_0, 1]\) such that \( x_2(t) \in A \) for all \( t \in [t_0, 1] \) and, from (29) and (39),

\[ d_{[t_0, 1]}(\hat{x}_2, \hat{x}) \leq [K^* + K_1(1 + 2\kappa K^*)] \left( 1 + |\ln(h(\hat{x}))| \right) h(\hat{x}). \quad (40) \]

From (30) and (38) we also have

\[ \hat{x}_2(t) = \vec{v} \quad \text{for a.e. } t \in \{ s \in [t_0, 1] : \hat{x}_2(s) \neq \hat{x}(s) \}. \]
Finally, invoking Lemma 4.6 in the appendix, we may replace the feasible \(coF\)-trajectory \(x_2(\cdot)\) on \([t_0, 1]\) by a feasible \(F\)-trajectory, for which the estimate (40) is preserved. Thus the assertion (5) of Theorem 2.1 is valid for \(x(\cdot)\) taken to be the ‘replaced’ \(x_2(\cdot)\), with \(K = K^* + K_1(1 + 2\kappa K^*)\).

The linear estimate is similarly obtained, in the strictly convex case.

We have proved the assertions of the theorem under the additional hypothesis (S1) and when \(\tilde{x}(t_0) = x_0 \in A\). Let us dispose of the latter condition. Fix \(K_0 > 0\), and take any \(t_0 \in [0, 1]\) and any \(F\)-trajectory \(\tilde{x}\) on \([t_0, 1]\) for which \(|\tilde{x}(t_0) - x_0| \leq K_0 h(\tilde{x})\). Define the \(F\)-trajectory \(\check{x}\) on \([t_0, 1]\):

\[
\check{x}(t) = \hat{x}(t) + (x_0 - \hat{x}(t_0)).
\]

We have that \(|\check{x} - \hat{x}|_{L^\infty} = |x_0 - \hat{x}(t_0)| \leq K_0 h(\hat{x})\). Moreover, since \(|b_1| = |b_2| = 1\),

\[
h(\check{x}) - h(\hat{x}) \leq K_0 h(\hat{x}) \tag{41}
\]

and \(\check{x}(t_0) \in A\). Hence we can find an \(F\)-trajectory \(x(\cdot)\) such that \(d_{[t_0, 1]}(\hat{x}, \check{x}) = d_{[t_0, 1]}(\hat{x}, \check{x}) \leq K_0 \ln(h(\hat{x}))\) for some \(K_0\) (independent of \(\hat{x}\)). Since \(h(\hat{x}) \leq \frac{1}{4}\) and in view of (41), this estimate remains valid (with a modified \(K_0\) which depends on \(K_0\)) when we replace \(\tilde{x}\) by \(\hat{x}\) on the right side. Similar considerations apply to the linear estimate (in the strictly convex case). So we have removed the extra condition (T1) on \(\hat{x}(t_0)\). Finally, we can remove also (S1), because the remaining hypotheses (H1) and (H2), together with the theorem assertions, are unaffected by a linear transformation of coordinates that replaces the linear independent vectors \(b_1\) and \(b_2\) by \(-e_1\) and \(-e_2\), respectively.

Appendix

**Lemma 4.5** Take a compact set \(F \subset \mathbb{R}^n\) with \(n \geq 2\). Let \(A\) be the set (4), in which \(b_1 = -e_1\) and \(b_2 = -e_2\). Assume that

\[
\text{co } F \cap \text{int } A \neq \emptyset. \tag{42}
\]

Then there exist numbers \(a_1 > 0\) and \(a_2 > 0\) with \(a_1^2 + a_2^2 = 1\), and a vector \(\tilde{v} \in \text{co } F\) such that \(\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)\) where \(\tilde{v}_1 > 0\) and \(\tilde{v}_2 > 0\) and, writing \(a = (a_1, a_2, 0, \ldots, 0)\), we have

\[
a \cdot \tilde{v} = \max_{v \in \text{co } F} a \cdot v.
\]

**Proof.** For \(\theta \in [0, \pi/2]\), consider the unit vector

\[
p(\theta) = (\cos \theta, \sin \theta, 0, \ldots, 0) \in \mathbb{R}^n. \tag{43}
\]

For any \(\theta \in [0, \pi/2]\) let \(W(\theta)\) be the set of support vectors of \(p(\theta)\) in \(\text{co } F\):

\[
W(\theta) = \left\{ v(\theta) \in \text{co } F : p(\theta) \cdot v(\theta) = \max_{w \in \text{co } F} p(\theta) \cdot w \right\}.
\]

We claim that there exists \(\bar{\theta} \in (0, \pi/2)\) and \(v(\bar{\theta}) \in W(\bar{\theta})\) such that

\[
\|v(\bar{\theta})\| \in \text{int}(A) = \left\{ x = (x_1, x_2, \ldots, x_n) : x_1 > 0, \ x_2 > 0 \right\}. \tag{44}
\]

This implies that the assertions of the lemma are satisfied with

\[
\tilde{v} = v(\bar{\theta}) \quad \text{and} \quad a = p(\bar{\theta}).
\]

To prove the claim, we make use of the fact that \(\theta \sim W(\theta)\) is a multifunction with closed graph, taking values (non-empty) convex, compact sets.
Invoking (42), we can find a vector \( \bar{w} = (\bar{w}_1, \ldots, \bar{w}_n) \in \text{co} F \cap \text{int} A \). Define \( \rho = \min \{ \bar{w}_1, \bar{w}_2 \} > 0 \). Then
\[
\max_{w \in \text{co} F} p(\theta) \cdot w \geq p(\theta) \cdot \bar{w} \geq \rho \quad \text{for all} \quad \theta \in [0, \pi/2].
\]
(45)

Now suppose the claim (44) were false. Then
\[
W(\theta) \cap \text{int}(A) = \emptyset \quad \text{for all} \quad \theta \in (0, \pi/2).
\]
(46)

In view of (45), this would imply that
\[
W(\theta) \subset C_1 \cup C_2 \quad \text{for all} \quad \theta \in (0, \pi/2)
\]
where
\[
C_1 \doteq \{ x \in \mathbb{R}^n : x_1 \geq \rho, x_2 \leq 0 \}
\]
and
\[
C_2 \doteq \{ x \in \mathbb{R}^n : x_1 \leq 0, x_2 \geq \rho \}.
\]
Since the multifunction \( \theta \mapsto W(\theta) \) has closed graph, we would also have
\[
W(0) \cap C_1 \neq \emptyset \quad \text{and} \quad W(\pi/2) \cap C_2 \neq \emptyset.
\]
Define
\[
\bar{\theta} = \sup \{ \theta \in (0, \pi/2) : W(\theta) \cap C_1 \neq \emptyset \}.
\]
Let \( \theta_i' \uparrow \bar{\theta} \) and \( \theta_i'' \downarrow \bar{\theta} \) be two sequences in \([0, \pi/2]\) such that
\[
W(\theta_i') \cap C_1 \neq \emptyset \quad \text{and} \quad W(\theta_i'') \cap C_2 \neq \emptyset
\]
and choose two sequences \( v_i' \) and \( v_i'' \) such that
\[
v_i' \in W(\theta_i') \cap C_1 \quad \text{and} \quad v_i'' \in W(\theta_i'') \cap C_2.
\]
By compactness we can extract two convergent subsequences \( v_{i_k}' \to v' \) and \( v_{i_k}'' \to v'' \). Since \( W(\cdot) \) has closed graph, we have
\[
v' \in W(\bar{\theta}) \cap C_1 \quad \text{and} \quad v'' \in W(\bar{\theta}) \cap C_2.
\]
But \( W(\bar{\theta}) \) is convex and so the segment \([v', v'']\) is contained in \( W(\bar{\theta}) \) and, since \( v' \in C_1 \) and \( v'' \in C_2 \), we conclude
\[
\text{int}(A) \cap [v', v''] \neq \emptyset.
\]
(47)

Note that \( \bar{\theta} \in (0, \pi/2) \) because \( \tan(\bar{\theta}) > 0 \). This fact, together with (47), contradicts (46).

We have verified the claim; the proof of the lemma is complete.

**Lemma 4.6** Let \( y : [0, 1] \to A \) be a \( \text{co} F \)-trajectory such that
\[
\dot{y}(t) = \bar{v} \in \text{co} F \cap \text{int} A \quad \text{for a.e.} \quad t \in J,
\]
and
\[
\dot{y}(t) \in F \quad \text{for a.e.} \quad t \in [0, 1] \setminus J,
\]
for some measurable set of times \( J \subseteq [0, 1] \). Then there exists an \( F \)-trajectory \( x : [0, 1] \to A \) such that
\[
x(0) = y(0), \quad \dot{x}(t) = \dot{y}(t) \quad \text{for a.e.} \quad t \notin J.
\]

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Proof. The set of times $\Omega = \{ t \in [0,1] \, ; \, y(t) \in \text{int}A \}$ is open and contains the measurable set $J$. We express $\Omega$ as a disjoint union of open intervals, say $\Omega = \bigcup_{\alpha} I_{\alpha}$, with $I_{\alpha} = [a_{\alpha}, b_{\alpha}[$. By assumption, $\bar{v} \in \text{co}F$, hence we can find vectors $v_0, v_1, \ldots, v_n \in F$ such that

$$\bar{v} = \sum_{i=0}^{n} \lambda_j v_j, \quad \lambda_j \geq 0, \quad \sum_{j=0}^{n} \lambda_j = 1.$$ 

Recalling that $\kappa = \max \{ |v| \, ; \, v \in F \}$ is an upper bound on the speed, we can further subdivide each interval $I_{\alpha}$ inserting an increasing set of points $\tau_{\alpha, \nu}$, $\nu \in \mathbb{Z}$, so that

$$d(x(\tau_{\alpha, \nu}), \partial A) > \kappa |\tau_{\alpha, \nu} - \tau_{\alpha, \nu-1}|.$$ (48)

Here $\partial A$ denotes the boundary of the cone $A$. For each fixed couple of indices $\alpha, \nu$ we now choose disjoint subsets $J_{\alpha, \nu}^{\ell} \subseteq J$, $\ell = 0, 1, \ldots, n$ such that

$$\text{meas}(J_{\alpha, \nu}^{\ell}) = \lambda_{\ell} \cdot \text{meas}(J \cap [\tau_{\alpha, \nu-1}, \tau_{\alpha, \nu}]).$$

The absolutely continuous function $x(\cdot)$ such that

$$x(0) = y(0), \quad \dot{x}(t) = \begin{cases} \dot{y}(t) & \text{if } t \notin J, \\ v_{\ell} & \text{if } t \in J_{\alpha, \nu}^{\ell} \text{ for some } \alpha, \nu, \text{ and some } \ell = 0, \ldots, n \end{cases}$$

satisfies all requirements. Indeed, our construction implies $x(t) = y(t)$ whenever $y(t) \in \partial A$ and also at all times $t = \tau_{\alpha, \nu}$. Moreover, for $t \in [\tau_{\alpha, \nu-1}, \tau_{\alpha, \nu}]$, by (48) one has

$$d(x(t), \partial A) \geq d(x(\tau_{\alpha, \nu}), \partial A) - \kappa |\tau_{\alpha, \nu} - \tau_{\alpha, \nu-1}| > 0.$$ 

Hence $x(t) \in A$ for all $t \in [0,1]$.

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References


